

# MODULI OF RIEMANN SURFACES, TRANSCENDENTAL ASPECTS

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These notes are an informal introduction to moduli spaces of compact Riemann surfaces via complex analysis, topology and Hodge Theory. The prerequisites for the first lecture are just basic complex variables, basic Riemann surface theory up to at least the Riemann-Roch formula, and some algebraic topology, especially covering space theory. Some good references for this material include [1] for complex analysis, [8] and [9] for the basic theory of Riemann surfaces, and [11] for algebraic topology. For later lectures I will assume more. The book by Clemens [5] and Chapter 2 of Griffiths and Harris [12] are excellent and are highly recommended. Other useful references include the surveys [16] and [14] and the book [17].

The first lecture covers moduli in genus 0 and genus 1 as these can be understood using relatively elementary methods, but illustrate many of the points which arise in higher genus. The notes cover more material than was covered in the lectures, and sometimes the order of topics in the notes differs from that in the lectures. I hope to add the material from the last lecture on the Torelli group and Morita's approach to the tautological classes in a future version.

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## LECTURE 1: LOW GENUS EXAMPLES

Suppose that  $g$  and  $n$  are non-negative integers. An  $n$ -pointed Riemann surface  $(C; x_1, \dots, x_n)$  of genus  $g$  is a compact Riemann surface  $C$  of genus  $g$  together with an ordered  $n$ -tuple of distinct points  $(x_1, \dots, x_n)$  of  $C$ . Two  $n$ -pointed Riemann surfaces  $(C; x_1, \dots, x_n)$  and  $(C'; x'_1, \dots, x'_n)$  are *isomorphic* if there is a biholomorphism  $f : C \rightarrow C'$  such that  $f(x_j) = x'_j$  when  $1 \leq j \leq n$ . The principal objects of study

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in these lectures are the spaces

$$\mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{isomorphism classes of } n\text{-pointed com-} \\ \text{pact Riemann surfaces } C \text{ of genus } g \end{array} \right\}$$

At the moment all we can say is that these are sets. One of the main objectives of these lectures is to show that each  $\mathcal{M}_{g,n}$  is a complex analytic variety with very mild singularities.

Later we will only consider  $\mathcal{M}_{g,n}$  when the stability condition

$$(1) \quad 2g - 2 + n > 0$$

is satisfied. But for the time being we will consider all possible values of  $g$  and  $n$ . When  $n = 0$ , we will simply write  $\mathcal{M}_g$  instead of  $\mathcal{M}_{g,0}$ .

The space  $\mathcal{M}_{g,n}$  is called the *moduli space of  $n$ -pointed curves (or Riemann surfaces) of genus  $g$* . The isomorphism class of  $(C; x_1, \dots, x_n)$  is called the *moduli point* of  $(C; x_1, \dots, x_n)$  and will be denoted by  $[C; x_1, \dots, x_n]$ .

There are (at least) two notions of the genus of a compact Riemann surface  $C$ . First there is the (analytic) genus

$$g(C) := \dim H^0(C, \Omega_C^1),$$

the dimension of the space of global holomorphic 1-forms on  $C$ . Second there is the *topological genus*

$$g_{\text{top}}(C) := \frac{1}{2} \text{rank } H_1(C, \mathbb{Z}).$$

Intuitively, this is the ‘number of holes’ in  $C$ . A basic fact is that these are equal. There are various ways to prove this, but perhaps the most standard is to use the Hodge Theorem (reference) which implies that

$$H^1(C, \mathbb{C}) \cong \{\text{holomorphic 1-forms}\} \oplus \{\text{anti-holomorphic 1-forms}\}.$$

The equality of  $g_{\text{top}}(C)$  and  $g(C)$  follows immediately as complex conjugation interchanges the holomorphic and antiholomorphic differentials.

Finally, we shall use the terms “complex curve” and “Riemann surface” interchangeably.

## 1. GENUS 0

It follows from Riemann-Roch formula that if  $X$  is a compact Riemann surface of genus 0, then  $X$  is biholomorphic to the Riemann sphere  $\mathbb{P}^1$ . So  $\mathcal{M}_0$  consists of a single point.

An *automorphism* of a Riemann surface  $X$  is simply a biholomorphism  $f : X \rightarrow X$ . The set of all automorphisms of  $X$  forms a group

$\text{Aut } X$ . The group  $GL_2(\mathbb{C})$  acts in  $\mathbb{P}^1$  via fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}$$

The scalar matrices  $S$  act trivially, and so we have a homomorphism

$$PGL_2(\mathbb{C}) \rightarrow \text{Aut } \mathbb{P}^1$$

where for any field  $F$

$$PGL_n(F) = GL_n(F)/\{\text{scalar matrices}\}$$

and

$$PSL_n(F) = SL_n(F)/\{\text{scalar matrices of determinant } 1\}.$$

**Exercise 1.1.** Prove that  $PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})$  and that these are isomorphic to  $\text{Aut } \mathbb{P}^1$ .

**Exercise 1.2.** Prove that  $\text{Aut } \mathbb{P}^1$  acts 3-transitively on  $\mathbb{P}^1$ . That is, given any two ordered 3-tuples  $(a_1, a_2, a_3)$  and  $(b_1, b_2, b_3)$  of distinct points of  $\mathbb{P}^1$ , there is an element  $f$  of  $\text{Aut } \mathbb{P}^1$  such that  $f(a_j) = b_j$  for  $j = 1, 2, 3$ . Show that  $f$  is unique.

**Exercise 1.3.** Prove that if  $X$  is a compact Riemann surface of genus 0, then  $X$  is biholomorphic to the Riemann sphere.

**Exercise 1.4.** Show that the automorphism group of an  $n$ -pointed curve of genus  $g$  is finite if and only if the stability condition (1) is satisfied. (Depending on what you know, you may find this a little difficult at present. More techniques will become available soon.)

Since  $\text{Aut } \mathbb{P}^1$  acts 3-transitively on  $\mathbb{P}^1$ , we have:

**Proposition 1.5.** *Every  $n$ -pointed Riemann surface of genus 0 is isomorphic to*

$$\begin{aligned} (\mathbb{P}^1; \infty) & \quad \text{if } n = 1; \\ (\mathbb{P}^1; 0, \infty) & \quad \text{if } n = 2; \\ (\mathbb{P}^1; 0, 1, \infty) & \quad \text{if } n = 3. \end{aligned}$$

□

**Corollary 1.6.** *If  $0 \leq n \leq 3$ , then  $\mathcal{M}_{0,n}$  consists of a single point.* □

The first interesting case is when  $n = 4$ . If  $(X; x_1, x_2, x_3, x_4)$  is a 4-pointed Riemann surface of genus 0, then there is a unique biholomorphism  $f : X \rightarrow \mathbb{P}^1$  with  $f(x_2) = 1$ ,  $f(x_3) = 0$  and  $f(x_4) = \infty$ . The value of  $f(x_1)$  is forced by these conditions. Since the  $x_j$  are distinct and  $f$  is a biholomorphism,  $f(x_1) \in \mathbb{C} - \{0, 1\}$ . It is therefore an invariant of  $(X; x_1, x_2, x_3, x_4)$ .

**Exercise 1.7.** Show that if  $g : X \rightarrow \mathbb{P}^1$  is any biholomorphism, then  $f(x_1)$  is the cross ratio

$$(g(x_1) : g(x_2) : g(x_3) : g(x_4))$$

of  $g(x_1), g(x_2), g(x_3), g(x_4)$ . Recall that the cross ratio of four distinct points  $x_1, x_2, x_3, x_4$  in  $\mathbb{P}^1$  is defined by

$$(x_1 : x_2 : x_3 : x_4) = \frac{(x_1 - x_3)/(x_2 - x_3)}{(x_1 - x_4)/(x_2 - x_4)}$$

The result of the previous exercise can be rephrased as a statement about moduli spaces:

**Proposition 1.8.** *The moduli space  $\mathcal{M}_{0,4}$  can be identified naturally with  $\mathbb{C} - \{0, 1\}$ . The moduli point  $[\mathbb{P}^1; x_1, x_2, x_3, x_4]$  is identified with the cross ratio  $(x_1 : x_2 : x_3 : x_4) \in \mathbb{C} - \{0, 1\}$ .  $\square$*

It is now easy to generalize this to general  $n \geq 4$ . Since every genus 0 Riemann surface is biholomorphic to  $\mathbb{P}^1$ , we need only consider  $n$ -pointed curves of the form  $(\mathbb{P}^1; x_1, \dots, x_n)$ . There is a unique automorphism  $f$  of  $\mathbb{P}^1$  such that  $f(x_1) = 0, f(x_2) = 1$  and  $f(x_3) = \infty$ . So every  $n$ -pointed Riemann surface of genus 0 is isomorphic to exactly one of the form

$$(\mathbb{P}^1; 0, 1, \infty, y_1, \dots, y_{n-3}).$$

To say that this is an  $n$ -pointed curve is to say that the points  $0, 1, \infty, y_1, \dots, y_{n-3}$  are distinct. That is, it

$$(y_1, \dots, y_{n-3}) \in (\mathbb{C} - \{0, 1\})^{n-3} - \Delta$$

where  $\Delta = \cup_{j < k} \Delta_{jk}$  is the union of the diagonals

$$\Delta_{jk} = \{(y_1, \dots, y_{n-3}) : y_j = y_k\}.$$

This is an affine algebraic variety as it is the complement of a divisor in an affine space. This shows that:

**Theorem 1.9.** *If  $n \geq 3$ , then  $\mathcal{M}_{0,n}$  is a smooth affine algebraic variety of dimension  $n - 3$  isomorphic to*

$$(\mathbb{C} - \{0, 1\})^{n-3} - \Delta.$$

The symmetric group  $\Sigma_n$  acts on  $\mathcal{M}_{0,n}$  by

$$\sigma : (\mathbb{P}^1; x_1, \dots, x_n) \mapsto (\mathbb{P}^1; x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

**Exercise 1.10.** Show that each  $\sigma \in \Sigma_n$  acts on  $\mathcal{M}_{0,n}$  as a regular mapping. (Hint: it suffices to consider the case of a transposition.)

**Exercise 1.11.** Suppose that  $n \geq 3$ . Construct a universal  $n$ -pointed genus 0 curve  $\mathcal{M}_{0,n} \times \mathbb{P}^1 \rightarrow \mathcal{M}_{0,n}$  that is equipped with  $n$  disjoint sections  $\sigma_1, \dots, \sigma_n$  such that

$$\sigma_j([\mathbb{P}^1; x_1, \dots, x_n]) = ([\mathbb{P}^1; x_1, \dots, x_n], x_j).$$

Show that it is universal in the sense that if  $f : X \rightarrow T$  is a family of smooth genus 0 curves over a smooth variety  $T$  and if the family has  $n$  sections  $s_1, \dots, s_n$  that are disjoint, then there is a holomorphic mapping  $\phi_f : T \rightarrow \mathcal{M}_{0,n}$  such that the pullback of the universal family is  $f$  and the pullback of  $\sigma_j$  is  $s_j$ .

## 2. GENUS 1

The study of the moduli space of genus 1 compact Riemann surfaces is very rich and has a long history because of its fundamental connections to number theory and the theory of plane cubic curves. We will take a transcendental approach to understanding  $\mathcal{M}_1$  which will reveal the connection with modular forms. Our first task is show that genus 1 Riemann surfaces can always be represented as the quotient of  $\mathbb{C}$  by a lattice.

One way to construct a Riemann surface of genus 1 is to take the quotient of  $\mathbb{C}$  by a lattice. Recall that a *lattice* in a finite dimensional real vector space  $V$  is a finitely generated (and therefore free abelian) subgroup  $\Lambda$  of  $V$  with the property that a basis of  $\Lambda$  as an abelian group is also a basis of  $V$  as a real vector space. A lattice in  $\mathbb{C}$  is thus a subgroup  $\Lambda$  of  $\mathbb{C}$  that is isomorphic to  $\mathbb{Z}^2$  and is generated by two complex numbers that are not real multiples of each other.

**Exercise 2.1.** Show that if  $\Lambda$  is a lattice in  $V$  and if  $\dim_{\mathbb{R}} V = d$ , then  $V/\Lambda$  is a compact manifold of real dimension  $d$ , which is diffeomorphic to the  $d$ -torus  $(\mathbb{R}/\mathbb{Z})^d$ .

If  $\Lambda$  is a lattice in  $\mathbb{C}$ , the quotient group  $\mathbb{C}/\Lambda$  is a compact Riemann surface which is diffeomorphic to the product of two circles, and so of genus 1.

**Theorem 2.2.** *If  $C$  is a compact Riemann surface of genus 1, then there is a lattice  $\Lambda$  in  $\mathbb{C}$  and an isomorphism  $\mu : C \rightarrow \mathbb{C}/\Lambda$ . If  $x_o \in C$ , then we may choose  $\mu$  such that  $\mu(x_o) = 0$ .*

The proof follows from the sequence of exercises below. Let  $C$  be a compact Riemann surface of genus 1.

**Exercise 2.3.** Show that a non-zero holomorphic differential on  $C$  has no zeros. Hint: Use Riemann-Roch.

Since  $g_{\text{top}}(C) = g(C) = 1$ , we know that  $H_1(C, \mathbb{Z})$  is free of rank 2. Fix a non-zero holomorphic differential  $w$  on  $C$ . Every other holomorphic differential is a multiple of  $w$ . The *period lattice* of  $C$  is defined to be

$$\Lambda = \left\{ \int_c w : c \in H_1(C, \mathbb{Z}) \right\}.$$

This is easily seen to be a subgroup of  $\mathbb{C}$ .

**Exercise 2.4.** Show that  $\Lambda$  is a lattice in  $\mathbb{C}$ . Hint: Choose a basis  $a, b$  of  $H_1(C, \mathbb{Z})$ . Show that if  $\int_a w$  and  $\int_b w$  are linearly independent over  $\mathbb{R}$ , then this would contradict the Hodge decomposition of  $H^1(C, \mathbb{C})$ .

Let  $E = \mathbb{C}/\Lambda$ . Our next task is to construct a holomorphic mapping from  $C$  to  $E$ .

**Exercise 2.5.** Fix a base point  $x_o$  of  $C$ . Define a mapping  $\nu : C \rightarrow E$  by

$$\nu(x) = \int_{\gamma} w$$

where  $\gamma$  is any smooth path in  $C$  that goes from  $x_o$  to  $x$ . Show that

- (i)  $\nu$  is well defined;
- (ii)  $\nu$  is holomorphic;
- (iii)  $\nu$  has nowhere vanishing differential, and is therefore a covering map;
- (iv) the homomorphism  $\nu_* : \pi_1(C, x_o) \rightarrow \pi_1(E, 0) \cong \Lambda$  is surjective, and therefore an isomorphism.

Deduce that  $\nu$  is a biholomorphism.

This completes the proof of Theorem 2.2. It has the following important consequence:

**Corollary 2.6.** *If  $C$  is a compact Riemann surface of genus 1, then the automorphism group of  $C$  acts transitively on  $C$ . Consequently, the natural mapping  $\mathcal{M}_{1,1} \rightarrow \mathcal{M}_1$  that takes  $[C; x]$  to  $[C]$  is a bijection.*

*Proof.* This follows as every genus 1 Riemann surface is isomorphic to one of the form  $\mathbb{C}/\Lambda$ . For such Riemann surfaces, we have the homomorphism

$$\mathbb{C}/\Lambda \rightarrow \text{Aut}(\mathbb{C}/\Lambda)$$

that takes the coset  $a + \Lambda$  to the translation  $z + \Lambda \mapsto z + a + \Lambda$ .  $\square$

**Definition 2.7.** An *elliptic curve* is a genus 1 curve  $C$  together with a point  $x_o \in C$ .

The previous result says that if  $C$  is a genus 1 curve and  $x_o$  and  $y_o$  are points of  $C$ , then the elliptic curves  $(C; x_o)$  and  $(C; y_o)$  are isomorphic.

The moduli space of elliptic curves is  $\mathcal{M}_{1,1}$ .

**Exercise 2.8.** Suppose that  $f : C \rightarrow \mathbb{C}/\Lambda$  is a holomorphic mapping from an arbitrary Riemann surface to  $\mathbb{C}/\Lambda$ . Let  $x_o$  be a base point of  $C$ . The 1-form  $dz$  on  $\mathbb{C}$  descends to a holomorphic differential  $w$  on  $\mathbb{C}/\Lambda$ . Its pullback  $f^*w$  is a holomorphic differential on  $C$ . Show that for all  $x \in C$ ,

$$f(x) = f(x_o) + \int_{\gamma} f^*w + \Lambda$$

where  $\gamma$  is a path in  $C$  from  $x_o$  to  $x$ .

**Exercise 2.9.** Use the results of the previous exercise to prove the following result.

**Corollary 2.10.** *If  $\Lambda_1$  and  $\Lambda_2$  are lattices in  $\mathbb{C}$ , then  $\mathbb{C}/\Lambda_1$  is isomorphic to  $\mathbb{C}/\Lambda_2$  if and only there exists  $\lambda \in \mathbb{C}^*$  such that  $\Lambda_1 = \lambda\Lambda_2$ .  $\square$*

**Exercise 2.11.** Show that if  $(C; x_o)$  is an elliptic curve, then  $C$  has a natural group structure with identity  $x_o$ .

We are finally ready to give a construction of  $\mathcal{M}_1$ . Recall that the complex structure on a Riemann surface  $C$  gives it a canonical orientation. This can be thought of as giving a direction of “positive rotation” about each point in the surface — the positive direction being that given by turning counter-clockwise about the point in any local holomorphic coordinate system. If  $C$  is compact, this orientation allows us to define the intersection number of two transversally intersecting closed curves on  $C$ . It depends only on the homology classes of the two curves and therefore defines the *intersection pairing*

$$\langle \ , \ \rangle : H_1(C, \mathbb{Z}) \otimes H_1(C, \mathbb{Z}) \rightarrow \mathbb{Z}.$$

If  $\alpha, \beta$  is a basis of  $H_1(C, \mathbb{Z})$ , then  $\langle \alpha, \beta \rangle = \pm 1$ . We shall call the basis *positive* if  $\langle \alpha, \beta \rangle = 1$ .

A *framing* of Riemann surface  $C$  of genus 1 is a positive basis  $\alpha, \beta$  of its first homology group. We will refer to  $(C : \alpha, \beta)$  as a *framed* genus 1 Riemann surface. Two framed genus 1 Riemann surfaces  $(C : \alpha, \beta)$  and  $(C' : \alpha', \beta')$  are isomorphic if there is a biholomorphism  $f : C \rightarrow C'$  such that  $\alpha' = f_*\alpha$  and  $\beta' = f_*\beta$ .

Let

$$\mathcal{X}_1 = \left\{ \begin{array}{l} \text{isomorphism classes of framed} \\ \text{Riemann surfaces of genus 1} \end{array} \right\}.$$

At the moment, this is just a set. But soon we will see that it is itself a Riemann surface. Note that forgetting the framing defines a function

$$\phi : \mathcal{X}_1 \rightarrow \mathcal{M}_1.$$

Denote the isomorphism class of  $(C : \alpha, \beta)$  by  $[C : \alpha, \beta]$ . If  $C$  is a genus 1 Riemann surface, then

$$\phi^{-1}([C]) = \{[C : \alpha, \beta] : (\alpha, \beta) \text{ is a positive basis of } H_1(C, \mathbb{Z})\}.$$

**Exercise 2.12.** Show that if  $(\alpha, \beta)$  and  $(\alpha', \beta')$  are two positive bases of  $H_1(C, \mathbb{Z})$ , then there is a unique element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of  $SL_2(\mathbb{Z})$  such that

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

(The reason for writing the basis vectors in the reverse order will become apparent shortly.)

Define an action of  $SL_2(\mathbb{Z})$  on  $\mathcal{X}_1$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [C : \alpha, \beta] = [C : \alpha', \beta']$$

where

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

**Exercise 2.13.** Show that there is a natural bijection

$$\mathcal{M}_1 \cong SL_2(\mathbb{Z}) \backslash \mathcal{X}_1.$$

At present,  $\mathcal{X}_1$  is just a set, but we now show that it is naturally a Riemann surface. We know from Theorem 2.2 that every element of  $\mathcal{X}_1$  is of the form  $[\mathbb{C}/\Lambda : \alpha, \beta]$ . But, by standard algebraic topology, there is a natural isomorphism

$$\Lambda \cong H_1(C, \mathbb{Z}).$$

Thus a basis of  $H_1(C, \mathbb{Z})$  corresponds to a basis of  $\Lambda$ .

**Exercise 2.14.** Suppose that  $\alpha, \beta$  is a basis of  $H_1(\mathbb{C}/\Lambda, \mathbb{Z})$  and that  $\omega_1, \omega_2$  is the corresponding basis of  $\Lambda$ . Show that  $\alpha, \beta$  is positive if and only if  $\omega_2/\omega_1$  has positive imaginary part.



It follows from this and Corollary 2.10 that

$$\mathcal{X}_1 = \{[\mathbb{C}/\Lambda : \omega_1, \omega_2] : \operatorname{Im}(\omega_2/\omega_1) > 0\}/\mathbb{C}^*$$

where the  $\mathbb{C}^*$ -action is defined by

$$\lambda \cdot [\mathbb{C}/\Lambda : \omega_1, \omega_2] = [\mathbb{C}/\lambda\Lambda : \lambda\omega_1, \lambda\omega_2].$$

We can go even further: since the basis  $\omega_1, \omega_2$  determines the lattice,

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2,$$

we can dispense with the lattice altogether. We have:

$$(2) \quad \mathcal{X}_1 = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \mathbb{C} \text{ and } \operatorname{Im}(\omega_2/\omega_1) > 0\}/\mathbb{C}^*$$

where  $\mathbb{C}^*$  acts on  $(\omega_1, \omega_2)$  by scalar multiplication.

Denote the upper half plane  $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$  by  $\mathbb{H}$ . Each  $\tau \in \mathbb{H}$  determines the element

$$[\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) : 1, \tau]$$

of  $\mathcal{X}_1$ . This defines a function  $\psi : \mathbb{H} \rightarrow \mathcal{X}_1$ . Under the identification (2),

$$\psi(\tau) = \text{the } \mathbb{C}^*\text{-orbit of } (1, \tau)$$

Since  $(\omega_1, \omega_2)$  and  $(1, \omega_2/\omega_1)$  are in the same orbit, we have proved:

**Theorem 2.15.** *The function  $\psi : \mathbb{H} \rightarrow \mathcal{X}_1$  is a bijection.*  $\square$

The group  $PSL_2(\mathbb{C})$  acts on  $\mathbb{P}^1$  by fractional linear transformations.

**Exercise 2.16.** Show that  $T \in PSL_2(\mathbb{C})$  satisfies  $T(\mathbb{H}) \subseteq \mathbb{H}$  if and only if  $T \in PSL_2(\mathbb{R})$ .

Thus the group  $SL_2(\mathbb{R})$  acts on  $\mathbb{H}$  by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

**Exercise 2.17.** Show that  $\psi : \mathbb{H} \rightarrow \mathcal{X}_1$  is  $SL_2(\mathbb{Z})$ -equivariant. That is, if  $T \in SL_2(\mathbb{Z})$ , then  $T\psi(\tau) = \psi(T\tau)$ .

**Theorem 2.18.** *There are natural bijections*

$$\mathcal{M}_1 \cong \mathcal{M}_{1,1} \cong SL_2(\mathbb{Z}) \backslash \mathbb{H}.$$

*Proof.* The first bijection was established in Corollary 2.6. The second follows from Exercise 2.13, Theorem 2.15 and Exercise 2.17.  $\square$

**Exercise 2.19.** Suppose that  $C$  is a genus 1 Riemann surface. Show that the point of  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$  that corresponds to  $[C] \in \mathcal{M}_1$  is the  $SL_2(\mathbb{Z})$  orbit of

$$\left( \int_{\beta} w / \int_{\alpha} w \right) \in \mathbb{H}$$

where  $w$  is any non-zero element of  $H^0(C, \Omega_C^1)$  and  $\alpha, \beta$  is a positive basis of  $H_1(C, \mathbb{Z})$ .

**2.1. Understanding  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ .** The Riemann surface structure on  $\mathbb{H}$  descends to a Riemann surface structure on  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ . Good references for this are Chapter VII of Serre's book [24], and Chapter 3 of Clemens' book [5]. We'll sketch part of the proof of the following fundamental theorem.

**Theorem 2.20.** *The quotient of  $\mathbb{H}$  by  $SL_2(\mathbb{Z})$  has a unique structure of a Riemann surface such that the projection  $\mathbb{H} \rightarrow SL_2(\mathbb{Z}) \backslash \mathbb{H}$  is holomorphic. Moreover, there is a biholomorphism between  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$  and  $\mathbb{C}$  which can be given by the modular function  $j : \mathbb{H} \rightarrow \mathbb{C}$ , where*

$$j(\tau) = \frac{1}{q} + 744 + 196\,884q + 21\,493\,760q^2 + \cdots$$

and  $q = e^{2\pi i \tau}$ .

The following exercises will allow you to construct most of the proof. The rest can be found in [24] and [5].

Let  $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ . This is a circle on the Riemann sphere which forms the boundary of  $\mathbb{H}$ . Let  $\overline{\mathbb{H}}$  be the closure of  $\mathbb{H}$  in the Riemann sphere  $\mathbb{P}^1$ ; it is the union of  $\mathbb{H}$  and  $\mathbb{P}^1(\mathbb{R})$ . Recall that every non-trivial element of  $PSL_2(\mathbb{C})$  has at most two fixed points in  $\mathbb{P}^1$ . Note that the fixed points of elements of  $PSL_2(\mathbb{R})$  are real or occur in complex conjugate pairs.

**Exercise 2.21.** Suppose that  $T \in SL_2(\mathbb{Z})$  is not a scalar matrix. Show that  $T$  has exactly

- (i) one fixed point in  $\mathbb{H}$  if and only if  $|\operatorname{tr} T| < 2$ ;
- (ii) one fixed point in  $\mathbb{P}^1(\mathbb{R})$  if and only if  $|\operatorname{tr} T| = 2$ ;
- (iii) two fixed points in  $\mathbb{P}^1(\mathbb{R})$  if and only if  $|\operatorname{tr} T| > 2$ .

Show that  $T \in SL_2(\mathbb{R})$  has finite order if and only if  $T$  has a fixed point in  $\mathbb{H}$ .

Fix an integer  $l \geq 0$ . The *level  $l$  subgroup* of  $SL_2(\mathbb{Z})$  is the subgroup of  $SL_2(\mathbb{Z})$  consisting of those matrices congruent to the identity mod  $l$ . We shall denote it by  $SL_2(\mathbb{Z})[l]$ . Since it is the kernel of the homomorphism  $SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/l)$ , it is normal and of finite index in  $SL_2(\mathbb{Z})$ .

**Exercise 2.22.** Show that  $SL_2(\mathbb{Z})[l]$  is torsion free for all  $l \geq 3$ . Hint: use the previous exercise. Deduce that  $SL_2(\mathbb{Z})[l] \backslash \mathbb{H}$  is a Riemann surface with fundamental group  $SL_2(\mathbb{Z})[l]$  and universal covering  $\mathbb{H}$  whenever  $l \geq 3$ .

The quotient of  $SL_2(\mathbb{Z})$  by its level  $l$  subgroup is  $SL_2(\mathbb{Z}/l)$ , which is a finite group. It follows that the projection

$$SL_2(\mathbb{Z})[l] \backslash \mathbb{H} \rightarrow SL_2(\mathbb{Z}) \backslash \mathbb{H}$$

is finite-to-one of degree equal to the half order of  $SL_2(\mathbb{Z}/l)$  when  $l > 2$ .

**Exercise 2.23.** Show that the quotient  $G \backslash C$  of a Riemann surface  $C$  by a finite subgroup  $G$  of  $\text{Aut } C$  has the structure of a Riemann surface such that the projection  $C \rightarrow G \backslash C$  is holomorphic. Hint: first show that for each point  $x$  of  $C$ , the isotropy group

$$G_x := \{g \in G : gx = x\}$$

is cyclic. This can be done by considering the actions of  $G_x$  on  $T_x X$  and  $\mathcal{O}_{X,x}$ .

This result, combined with the previous exercises, establishes that  $\mathcal{M}_1 = SL_2(\mathbb{Z}) \backslash \mathbb{H}$  has a natural structure of a Riemann surface such that the projection  $\mathbb{H} \rightarrow \mathcal{M}_1$  is holomorphic.

Serre [24, p. 78] proves that a fundamental domain for the action of  $SL_2(\mathbb{Z})$  on  $\mathbb{H}$  is the region

$$F = \{\tau \in \mathbb{H} : |\text{Re } \tau| \leq 1/2, |\tau| \geq 1\}.$$

Points of  $F$  can be thought of as giving a canonical framing of a lattice

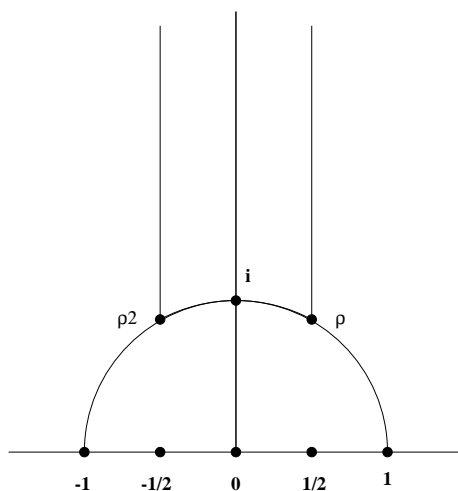


FIGURE 1. A a fundamental domain of  $SL_2(\mathbb{Z})$  in  $\mathbb{H}$

$\Lambda$  in  $\mathbb{C}$ . Such a framing is given as follows: the first basis element is a non-zero vector of shortest length in  $\Lambda$ , the second basis element is a shortest vector in  $\Lambda$  that is not a multiple of the first.

**Exercise 2.24.** Show that  $\tau \in F$  if and only if  $(1, \tau)$  is such a canonical basis of the lattice  $\mathbb{Z} \oplus \mathbb{Z}\tau$ .

Serre [24, p. 78] also proves that  $PSL_2(\mathbb{Z})$  is generated by  $\tau \mapsto -1/\tau$  and  $\tau \mapsto \tau + 1$ .

**Exercise 2.25.** Use this to prove that  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$  is the quotient of the fundamental domain  $F$  obtained by identifying the opposite vertical sides, and by identifying the arc of the circle  $|\tau| = 1$  from  $\rho$  to  $i$  with the arc from  $\rho^2$  to  $i$ . Deduce that  $\mathcal{M}_1 = SL_2(\mathbb{Z}) \backslash \mathbb{H}$  is a Riemann surface homeomorphic to a disk.

It is conceivable that  $\mathcal{M}_1$  is biholomorphic to a disk, for example. But this is not the case as  $\mathcal{M}_1$  can be compactified by adding one point.

**Exercise 2.26.** Show that  $\mathcal{M}_1$  can be compactified by adding a single point  $\infty$ . A coordinate neighbourhood of  $\infty$  is the unit disk  $\Delta$ . Denote the holomorphic coordinate in it by  $q$ . The point  $\tau$  of  $\mathcal{M}_1 = SL_2(\mathbb{Z}) \backslash \mathbb{H}$  is identified with the point  $e^{2\pi i\tau}$  of  $\Delta$ . Show that

$$\mathcal{M}_1 \cup \{\infty\}$$

is a compact Riemann surface of genus 0 where  $q$  is a local parameter about  $\infty$  and where  $\mathcal{M}_1$  holomorphically embedded. Deduce that  $\mathcal{M}_1$  is biholomorphic to  $\mathbb{C}$ .

*Remark 2.27.* This compactification is the moduli space  $\overline{\mathcal{M}}_{1,1}$ .

Since every compact Riemann surface is canonically a complex algebraic curve, this shows that  $\mathcal{M}_1$  is an algebraic variety.

**2.2. Automorphisms.** The automorphisms of an elliptic curve are intimately related with the set of elements of  $SL_2(\mathbb{Z})$  that stabilize the points corresponding to it in  $\mathbb{H}$ .

**Exercise 2.28.** Suppose that  $C$  is a genus 1 Riemann surface curve. Suppose that  $\phi : C \rightarrow C$  is an automorphism of  $C$ . Show that

- (i)  $\phi$  is a translation if and only if  $\phi_* : H_1(C, \mathbb{Z}) \rightarrow H_1(C, \mathbb{Z})$  is the identity;
- (ii) if  $\alpha, \beta$  is a positive basis of  $H_1(C, \mathbb{Z})$ , then

$$[C : \alpha, \beta] = [C : \phi_*\alpha, \phi_*\beta] \in \mathcal{X}_1;$$

Deduce that there is a natural isomorphism

$$\text{Aut}(C; x_o) \cong \text{stabilizer in } SL_2(\mathbb{Z}) \text{ of } [C : \alpha, \beta] \in \mathcal{X}_1.$$

Note that any two points of  $\mathbb{H}$  that lie in the same orbit of  $SL_2(\mathbb{Z})$  have isomorphic stabilizers. Consequently, to find all elliptic curves with automorphism groups larger than  $\mathbb{Z}/2\mathbb{Z}$ , one only has to look for points in the fundamental domain with stabilizers larger than  $\mathbb{Z}/2\mathbb{Z}$ .

**Exercise 2.29.** Show that the stabilizer in  $SL_2(\mathbb{Z})$  of  $\tau \in \mathbb{H}$  is

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} & \tau \notin \text{orbit of } i \text{ and } \rho; \\ \mathbb{Z}/4\mathbb{Z} & \tau \in \text{orbit of } i; \\ \mathbb{Z}/6\mathbb{Z} & \tau \in \text{orbit of } \rho. \end{cases}$$

An immediate consequence of this computation is the following:

**Theorem 2.30.** *If  $(C; x_o)$  is an elliptic curve, then*

$$\text{Aut}(C, x_o) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } (C; x_o) \cong (\mathbb{C}/\mathbb{Z}[i]; 0) \\ \mathbb{Z}/6\mathbb{Z} & \text{if } (C; x_o) \cong (\mathbb{C}/\mathbb{Z}[\rho]; 0) \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \end{cases}$$

It is easy to see from this, for example, that the Fermat cubic

$$x^3 + y^3 + z^3 = 0$$

in  $\mathbb{P}^2$  has automorphism group  $\mathbb{Z}/6\mathbb{Z}$  and therefore is isomorphic to  $\mathbb{Z}/\mathbb{Z}[\rho]$ .

**2.3. Families of Genus 1 and Elliptic Curves.** Suppose that  $X$  is a complex analytic manifold and that  $f : X \rightarrow T$  is a holomorphic mapping to another complex manifold each of whose fibers is a genus 1 curve. We say that  $f$  is a family of genus 1 curves. If there is a section  $s : T \rightarrow X$  of  $f$ , then the fiber of  $f$  over  $t \in T$  is an elliptic curve with identity  $s(t)$ . We say that  $f$  is a family of elliptic curves.

Each such family gives rise to a function

$$\phi_f : T \rightarrow \mathcal{M}_1$$

that is defined by taking  $t \in T$  to the moduli point  $[X_t]$  of the fiber  $X_t$  of  $f$  over  $t$ . We shall call it the *period mapping* of the family.

**Theorem 2.31.** *The period map  $\phi_f$  is holomorphic.*

*Sketch of Proof.* For simplicity, we suppose that  $T$  has complex dimension 1. The relative holomorphic tangent bundle of  $f$  is the holomorphic line bundle  $T'_f$  on  $X$  consisting of holomorphic tangent vectors to  $X$  that are tangent to the fibers of  $f$ . That is,

$$T'_f = \ker\{T'X \rightarrow T'T\}.$$

The sheaf of holomorphic sections of its dual is called the relative dualizing sheaf and is often denoted by  $\omega_{X/T}$ . The push forward  $f_*\omega_{X/T}$  of this sheaf to  $T$  is a holomorphic line bundle over  $T$  and has fiber

$$H^0(X_t, \Omega_{X_t}^1)$$

over  $t \in T$ . Fix a reference point  $t_o \in T$ . Let  $w(t)$  be a local holomorphic section of  $\omega_{X/T}$  defined in a contractible neighbourhood  $U$  of  $t_o$ . Since bundle is locally topologically trivial,  $f^{-1}(U)$  is homeomorphic to  $U \times X_{t_o}$ . We can thus identify  $H_1(X_t, \mathbb{Z})$  with  $H_1(X_{t_o}, \mathbb{Z})$  for each  $t \in U$ . Fix a positive basis  $\alpha, \beta$  of  $H_1(X_{t_o}, \mathbb{Z})$ . This can be viewed as a positive basis of  $H_1(X_t, \mathbb{Z})$  for all  $t \in U$ .

It is not difficult to show that

$$\int_{\alpha} w(t) \text{ and } \int_{\beta} w(t)$$

vary holomorphically with  $t \in U$ . It follows that

$$\tau(t) := \left\{ \int_{\beta} w(t) \middle/ \int_{\alpha} w(t) \right\}$$

varies holomorphically with  $t \in U$ . This shows that map  $\phi_f$  is holomorphic in the neighbourhood  $U$  of  $t_o$ . It follows that  $\phi_f$  is holomorphic.  $\square$

If you examine the proof, you will see that we really proved two extra facts. First, the period mapping  $\phi_f : T \rightarrow \mathcal{M}_1$  associated to every family  $f : X \rightarrow T$  of genus 1 curves is *locally liftable* to a holomorphic mapping to  $\mathbb{H}$ . If, in addition,  $f$  is a family of elliptic curves, then the period mapping  $\phi_f$  determines the family  $f$  and the section  $s$ :

**Proposition 2.32.** *The period mapping  $\phi_f$  associated to a family  $f : X \rightarrow T$  of genus 1 curves is locally liftable to a holomorphic mapping to  $\mathbb{H}$ . If  $f$  is a family of elliptic curves, then  $\phi_f$  can be globally lifted to a holomorphic mapping  $\tilde{\phi}_f : \tilde{T} \rightarrow \mathbb{H}$  and there is a homomorphism  $\phi_{f*} : \pi_1(T, t_o) \rightarrow SL_2(\mathbb{Z})$  (unique up to conjugacy) such that the diagram*

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{\phi}_f} & \mathbb{H} \\ \downarrow & & \downarrow \\ T & \xrightarrow[\phi_f]{} & \mathcal{M}_1 \end{array}$$

*commutes and such that*

$$\tilde{\phi}_f(\gamma \cdot x) = \phi_{f*}(\gamma) \cdot \tilde{\phi}_f(x)$$

*for all  $x \in \tilde{T}$  and all  $\gamma \in \pi_1(T, t_o)$ .*  $\square$

The proof is left as an exercise. We shall see in a moment that the converse of this result is also true.

This result has an important consequence — and that is that not every holomorphic mapping  $T \rightarrow \mathcal{M}_1$  is the period mapping of a holomorphic family of elliptic (or even genus 1) curves. The reason for this is that not every mapping  $T \rightarrow \mathcal{M}_1$  is locally liftable.

**Exercise 2.33.** Show that the identity mapping  $\mathcal{M}_1 \rightarrow \mathcal{M}_1$  is not locally liftable. Deduce that there is no family of genus 1 curves over  $\mathcal{M}_1$  whose period mapping is the identity mapping. In particular, show that there is no universal elliptic curve over  $\mathcal{M}_1$ .

To each genus 1 curve  $C$ , we can canonically associate the elliptic curve  $\text{Jac } C := \text{Pic}^0 C$  which we shall call the *jacobian* of  $C$ . Abel's Theorem tells us that  $C$  and  $\text{Jac } C$  are isomorphic as genus 1 curves, but the isomorphism depends on the choice of a base point of  $C$ .

**Exercise 2.34.** Show that for each family  $f : X \rightarrow T$  of genus 1 curves the corresponding family of jacobians is a family of elliptic curves. Show that these two families have the same period mapping. Show that if  $f$  is a family of elliptic curves, then the family of jacobians is canonically isomorphic to the original family  $f$ .

### 3. ORBIFOLDS

The discussion in the previous section suggests that  $\mathcal{M}_1$  should be viewed as  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$  rather than as  $\mathbb{C}$  — or that  $\mathcal{M}_1$  is not an algebraic variety or a manifold, but rather something whose local structure includes the information of how it is locally the quotient of a disk by a finite group. In topology such objects are called *orbifolds* and in algebraic geometry *stacks*. Very roughly speaking, orbifolds are to manifolds as stacks are to varieties. The moduli spaces  $\mathcal{M}_{g,n}$  are often conveniently viewed as orbifolds or as stacks.

For us, an orbifold is a topological space that is the quotient of a simply connected topological space  $X$  by a group  $\Gamma$ . The group is required to act properly discontinuously on  $X$  and all isotropy groups are required to be finite.<sup>1</sup> For example,  $\mathcal{M}_1$  can be viewed as an orbifold as the quotient of  $\mathbb{H}$  by  $SL_2(\mathbb{Z})$ .

This definition is not as general as it could be, but since all of our moduli spaces  $\mathcal{M}_{g,n}$  are of this form, it is good enough for our purposes.

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<sup>1</sup>We could have added some other natural conditions, such as requiring that there be a finite index subgroup of  $\Gamma$  that acts on  $X$  fixed point freely. We could also require that the set of elements of  $G$  that act trivially on  $X$  is central in  $\Gamma$ . Both of these conditions are natural and are satisfied by all of our primary examples of orbifolds, the  $\mathcal{M}_{g,n}$ .

The general definition is obtained by “sheafifying” this one — i.e., general orbifolds are locally the quotient of a simply connected space by a finite group. A general definition along these lines can be found in Chapter 13 of [25]. Mumford’s definition of stacks can be found in [23].

A morphism  $f : \Gamma_1 \backslash X_1 \rightarrow \Gamma_2 \backslash X_2$  of orbifolds is a continuous map that arises as follows: there is a homomorphism  $f_* : \Gamma_1 \rightarrow \Gamma_2$  and a continuous mapping  $\tilde{f} : X_1 \rightarrow X_2$  that is equivariant with respect to  $f_*$ ; that is, the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\tilde{f}} & X_2 \\ g \downarrow & & \downarrow f_*(g) \\ X_1 & \xrightarrow{\tilde{f}} & X_2 \end{array}$$

commutes for all  $g \in \Gamma_1$ .

At this stage, I should point out that every reasonable topological space  $X$  can be viewed as the quotient  $\pi_1(X, x) \backslash \tilde{X}$  of its universal covering by its fundamental group. In this way, every topological space can be regarded as an orbifold. It thus it makes sense to talk about orbifold mappings between orbifolds and ordinary topological spaces.

An orbifold  $\Gamma \backslash X$  can have an enriched structure — such as a smooth, riemannian, Kähler, or algebraic structure. One just insists that its “orbifold universal covering”  $\tilde{X}$  has such a structure and that  $\Gamma$  act on  $\tilde{X}$  as automorphisms of this structure. Maps between two orbifolds with the same kind of enriched structure are defined in the obvious way. For example, a map between two orbifolds with complex structures is given by an equivariant holomorphic map between their orbifold universal coverings.

Many orbifolds are given as quotients of non-simply connected spaces by a group that acts discontinuously, but not fixed point freely. Such quotients have canonical orbifold structures: If  $M$  is a topological space and  $G$  a group that acts discontinuously on  $M$ , then

$$G \backslash M \cong \Gamma \backslash \tilde{M}$$

where  $p : \tilde{M} \rightarrow M$  is the universal covering<sup>2</sup> of  $M$  and

$$\Gamma = \{(\phi, g) : \text{where } \phi : \tilde{M} \rightarrow \tilde{M} \text{ covers } g \in G\}.$$

---

<sup>2</sup>Of course, I am assuming that  $M$  is nice enough as a topological space to have a universal covering. We assume, for example, that  $M$  is locally simply connected, a condition satisfied by all complex algebraic varieties and all manifolds.



Here  $\phi$  covers  $g \in G$  means that the diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\phi} & \widetilde{M} \\ p \downarrow & & \downarrow p \\ M & \xrightarrow{g} & M \end{array}$$

commutes.

There is a notion of the *orbifold fundamental* group  $\pi_1^{\text{orb}}(\Gamma \backslash X, x_o)$  of a connected pointed orbifold. It is a variant of the definition of the fundamental group of a pointed topological space.

First, we'll fix the convention that the composition  $\alpha\beta$  of two paths  $\alpha$  and  $\beta$  in a space is defined when  $\alpha(1) = \beta(0)$ . The product path is the one obtained by traversing  $\alpha$  first, then  $\beta$ . This is the convention used in [11], for example, and is the opposite of the convention used by many algebraic geometers such as Deligne.

We need to impose several conditions on  $X$  and on  $x_o$  in order for the definition to make sense. Let  $p : X \rightarrow \Gamma \backslash X$  be the projection. Note that if  $x, y \in p^{-1}(x_o)$ , then the isotropy groups  $\Gamma_y$  and  $\Gamma_z$  are conjugate in  $\Gamma$ . Our first assumption is that  $\Gamma_y$  is contained in the center of  $\Gamma$  for one (and hence all)  $y \in p^{-1}(x_o)$ . For such  $x_o$ , we shall denote the common isotropy group of the points lying over  $x_o$  by  $\Gamma_{x_o}$ . Our second assumption is that  $X$  is connected, locally path connected, and locally simply connected. These conditions are satisfied by all connected complex algebraic varieties. Finally, we shall assume that if  $g \in \Gamma$  acts trivially in the neighbourhood of some point of  $X$ , it acts trivially on all of  $X$ . All three of these conditions are natural and will be satisfied in cases of interest to us.

Let

$$P(x_o) = \{\text{homotopy classes of paths } ([0, 1], \{0, 1\}) \rightarrow (X, p^{-1}(x_o))\}.$$

Since  $\Gamma$  acts on the left of  $X$ , this is a left  $\Gamma$ -set. Let

$$Q(x_o) = \{(g, \gamma) \in \Gamma \times P(x_o) : g^{-1} \cdot \gamma(0) = \gamma(1)\}.$$

This has a natural left  $\Gamma$ -action given by

$$g : (h, \gamma) \mapsto (ghg^{-1}, g \cdot \gamma).$$

Define  $\pi_1^{\text{orb}}(\Gamma \backslash X, x_o)$  to be the quotient  $\Gamma \backslash Q(x_o)$ . This has a natural group structure which can be understood by noting that  $Q(x_o)$  is a groupoid. The composition of two elements  $(g, \gamma)$  and  $(h, \mu)$  is defined when  $\gamma(1) = \mu(0)$ . It is then given by

$$(g, \gamma) \cdot (h, \mu) = (gh, \gamma\mu).$$

To multiply two elements of  $\pi_1^{\text{orb}}(G \backslash X, x_o)$ , translate one of them until they are composable. For example, if we fix a point  $\tilde{x}_o$  of  $p^{-1}(x_o)$ , then each element of  $\pi_1^{\text{orb}}(G \backslash X, x_o)$  has a representative of the form  $(g, \gamma)$  where  $\gamma(0) = \tilde{x}_o$ . This representation is unique up to translation by an element of  $\Gamma_{x_o}$ . To multiply two elements  $(g, \gamma)$  and  $(h, \mu)$  starting at  $\tilde{x}_o$ , multiply  $(h, \mu)$  by  $g$  so that it can be composed with  $(g, \gamma)$ . Then the product of these two elements in  $\pi_1^{\text{orb}}(G \backslash X, x_o)$  is represented by the path

$$(g, \gamma) \cdot (g^{-1}hg, g^{-1} \cdot \mu) = (hg, \gamma(g^{-1} \cdot \mu)).$$

**Exercise 3.1.** Show that if  $k \in \Gamma_{x_o}$ , then  $k$  acts trivially on  $P(x_o)$ . Deduce that the multiplication above is well defined. (This is where we need  $\Gamma_{x_o}$  to be central in  $\Gamma$ .)

The following exercises should help give some understanding of orbifold fundamental groups.

**Exercise 3.2.** Show that if  $\Gamma$  acts fixed point freely on  $X$ , then there is a natural isomorphism between  $\pi_1^{\text{orb}}(\Gamma \backslash X, x_o)$  and the usual fundamental group  $\pi_1(\Gamma \backslash X, x_o)$ . In particular, if we view a topological space  $X$  as an orbifold, then the two notions of fundamental group agree.

**Exercise 3.3.** Show that if  $\Gamma$  is an abelian group that acts trivially on a one point space  $X = \{*\}$ , then  $\pi_1^{\text{orb}}(\Gamma \backslash X, x_o)$  is defined and there is a natural isomorphism

$$\pi_1^{\text{orb}}(\Gamma \backslash X, x_o) \cong \Gamma.$$

**Exercise 3.4.** Let  $X = \mathbb{C}$  and  $\Gamma = \mathbb{Z}/n\mathbb{Z}$ . Define an action  $\Gamma$  on  $X$  by letting the generator 1 of  $\mathbb{Z}/n\mathbb{Z}$  act by multiplication by  $e^{2\pi i/n}$ . Show that  $\pi_1^{\text{orb}}(\Gamma \backslash X, x_o)$  is defined for all  $x_o \in \Gamma \backslash \mathbb{C}$ , and for each  $x_o$  there is a natural isomorphism

$$\pi_1^{\text{orb}}(\Gamma \backslash X, x_o) \cong \mathbb{Z}/n\mathbb{Z}.$$

**Exercise 3.5.** Show that for each choice of a point  $x \in p^{-1}(x_o)$ , there is a natural isomorphism

$$\phi_x : \pi_1^{\text{orb}}(\Gamma \backslash X, x_o) \rightarrow \Gamma$$

defined by  $\phi_x(g, \gamma) = g^{-1}$  when  $\gamma(0) = x$ . Show that if  $y = hx$ , then  $\phi_y = h\phi_x h^{-1}$ .

*Remark 3.6.* One can also define the orbifold fundamental group of  $\Gamma \backslash X$  using the Borel construction. First, find a contractible space  $E\Gamma$  on which  $\Gamma$  acts discontinuously and fixed point freely (any one will

do). There is a canonical construction of such spaces. (See for example [3, p. 19].) Fix a base point  $e_o \in E\Gamma$ . The diagonal action of  $\Gamma$  on  $E\Gamma \times X$ , a simply connected space, is fixed point free and the map

$$q : E\Gamma \times X \rightarrow \Gamma \backslash (E\Gamma \times X)$$

is a covering mapping with Galois group  $\Gamma$ . For  $x \in X$ , define

$$\pi_1^{\text{orb}}(\Gamma \backslash X, q(e_o, x)) = \pi_1(\Gamma(E\Gamma \times X), q(e_o, x)).$$

If  $\Gamma_x$  is central in  $\Gamma$ , then this depends only on  $x_o = p(x)$ . It is not difficult to show that, in this case, this definition agrees with the more elementary one given above.

The moduli space  $\mathcal{M}_1$  is naturally an orbifold, being the quotient of  $\mathbb{H}$  by  $SL_2(\mathbb{Z})$ . The condition on  $x_o$  is satisfied for all points not in the orbit of  $i$  or  $\rho$ . Consequently,  $\pi_1^{\text{orb}}(\mathcal{M}_1, x_o)$  is defined for all  $x_o$  other than those corresponding to the orbits of  $i$  and  $\rho$ , and for each such  $x_o$ , there is an isomorphism

$$\pi_1^{\text{orb}}(\mathcal{M}_1, x_o) \cong SL_2(\mathbb{Z})$$

which is well defined up to conjugacy.

This gives the following restatement of Proposition 2.32.

**Proposition 3.7.** *If  $f : X \rightarrow T$  is a holomorphic family of genus 1 curves, then the period mapping  $\phi_f : T \rightarrow \mathcal{M}_1$  is a morphism of orbifolds.*

*Remark 3.8.* Since  $\mathcal{M}_1$  can also be written as  $PSL_2(\mathbb{Z}) \backslash \mathbb{H}$ , it is natural to ask why we are not giving  $\mathcal{M}_1$  this orbifold structure. This question will be fully answered in the section on the universal elliptic curve and in subsequent sections on curves of higher genus. For the time being, just note that if we give  $\mathcal{M}_1$  the orbifold structure  $PSL_2(\mathbb{Z}) \backslash \mathbb{H}$ , then  $\pi^{\text{orb}}(\mathcal{M}_1)$  would be  $PSL_2(\mathbb{Z})$  instead of  $SL_2(\mathbb{Z})$ .

**3.1. The Universal Elliptic Curve.** Let's attempt to construct a "universal elliptic curve" over  $\mathcal{M}_1$ . We begin by constructing one over  $\mathbb{H}$ . The group  $\mathbb{Z}^2$  acts on  $\mathbb{C} \times \mathbb{H}$  by

$$(n, m) : (z, \tau) \mapsto (z + n\tau + m, \tau).$$

This action is fixed point free, so the quotient  $\mathbb{Z}^2 \backslash (\mathbb{C} \times \mathbb{H})$  is a complex manifold. The fiber of the projection

$$\mathbb{Z}^2 \backslash (\mathbb{C} \times \mathbb{H}) \rightarrow \mathbb{H}$$

over  $\tau$  is simply the elliptic curve  $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ . The family has the section that takes  $\tau \in \mathbb{H}$  to the identity element of the fiber lying over it. So this really is a family of elliptic curves.

Let's see what happens if we try to quotient out by  $\mathbb{Z}^2$  and  $SL_2(\mathbb{Z})$  at the same time. First note that  $SL_2(\mathbb{Z})$  acts on  $\mathbb{Z}^2$  on the right by matrix multiplication. We can thus form the semi-direct product  $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ . This is the group whose underlying set is  $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$  and whose multiplication is given by

$$(A_1, (n_1, m_1))(A_2, (n_2, m_2)) = (A_1 A_2, (n_1, m_1)A_2 + (n_2, m_2)).$$

**Exercise 3.9.** Show that the action of  $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$  on  $\mathbb{C} \times \mathbb{H}$  given by

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (n, m) \right) : (z, \tau) \mapsto \left( \frac{z + n\tau + m}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right)$$

is well defined. Show that there is a well defined projection

$$(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \backslash (\mathbb{C} \times \mathbb{H}) \rightarrow SL_2(\mathbb{Z}) \backslash \mathbb{H}.$$

This is a reasonable candidate for the universal curve. But we should be careful.

**Exercise 3.10.** Show that the fiber of the natural projection

$$(3) \quad (SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \backslash (\mathbb{C} \times \mathbb{H}) \rightarrow SL_2(\mathbb{Z}) \backslash \mathbb{H}$$

over the point corresponding to the elliptic curve  $(C; 0)$  is the quotient of  $C$  by the finite group  $\text{Aut}(C; 0)$ . In particular, the fiber of the projection over  $[(C; 0)]$  is always a quotient of  $C/\pm \cong \mathbb{P}^1$ , and is never  $C$ .

However, it is more natural to regard  $\mathcal{E} := (SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \ltimes \mathbb{H}$  as an orbifold. We shall do this. First note that the projection (3) has an orbifold section that is induced by the mapping

$$\mathbb{H} \rightarrow \mathbb{C} \times \mathbb{H}$$

that takes  $\tau$  to  $(0, \tau)$ . Note that if we consider  $\mathcal{M}_1$  as the orbifold  $PSL_2(\mathbb{Z}) \backslash \mathbb{H}$ , then the section would not exist. It is for this reason that we give  $\mathcal{M}_1$  the orbifold structure with orbifold fundamental group  $SL_2(\mathbb{Z})$  instead of  $PSL_2(\mathbb{Z})$ .

The following should illustrate why it is more natural to view  $\mathcal{M}_1$  as an orbifold than as a variety.

**Theorem 3.11.** *There is a natural one-to-one correspondence between holomorphic orbifold mappings from a smooth complex curve (or variety)  $T$  to  $\mathcal{M}_1$  and families of elliptic curves over  $T$ . The correspondence is given by pullback.  $\square$*

We could define a ‘geometric points’ of an orbifold  $X$  to be an orbifold map from a point with trivial fundamental group to  $X$ . For example, suppose  $\tau$  is any point of  $\mathbb{H}$ . Denote its isotropy group in  $SL_2(\mathbb{Z})$  by  $\Gamma_\tau$ . Then  $\Gamma_\tau \backslash \{\tau\}$  is a point of  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ , but is not a geometric point as it has fundamental group  $\Gamma_\tau$ , which is always non-trivial as it contains  $\mathbb{Z}/2\mathbb{Z}$ . The corresponding geometric point corresponds to the ‘universal covering’  $\{\tau\} \rightarrow \Gamma_\tau \backslash \{\tau\}$  of this point. One can define pullbacks of orbifolds in such a way that the fiber of the pullback of the universal curve to the geometric point  $\tau$  of  $\mathcal{M}_1$  is the corresponding elliptic curve  $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ .

Denote the  $n$ -fold fibered product of  $\mathcal{E} \rightarrow \mathcal{M}_1$  with itself by  $\mathcal{E}^{(n)} \rightarrow \mathcal{M}_1$ . It is naturally an orbifold (exercise). This has divisors  $\Delta_{jk}$  that consist of the points of  $\mathcal{E}^{(n)}$  where the  $j$ th and  $k$ th points agree. It also has divisors  $\Delta_j$  where the  $j$ th point is zero.

**Exercise 3.12.** Show that the moduli space  $\mathcal{M}_{1,n}$  can be identified with

$$\mathcal{E}^{(n-1)} - \left( \bigcup_{j < k} \Delta_{jk} \cup \bigcup_j \Delta_j \right).$$

**3.2. Modular Forms.** There is a natural orbifold line bundle  $\mathcal{L}$  over  $\mathcal{M}_1$ . It is the quotient of  $\mathbb{H} \times \mathbb{C}$  by  $SL_2(\mathbb{Z})$  where  $SL_2(\mathbb{Z})$  acts via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, z) \rightarrow \left( \frac{a\tau + b}{c\tau + d}, (c\tau + d)z \right).$$

**Exercise 3.13.** Show that this is indeed an action. Show that the  $k$ th power  $\mathcal{L}^{\otimes k}$  of this line bundle is the quotient of  $\mathbb{H} \times \mathbb{C}$  by the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, z) \rightarrow \left( \frac{a\tau + b}{c\tau + d}, (c\tau + d)^k z \right).$$

Orbifold sections of  $\mathcal{L}^{\otimes k}$  correspond to holomorphic functions  $f : \mathbb{H} \rightarrow \mathbb{C}$  for which the mapping

$$\mathbb{H} \rightarrow \mathbb{H} \times \mathbb{C} \quad \tau \mapsto (\tau, f(\tau))$$

is  $SL_2(\mathbb{Z})$ -equivariant.

**Exercise 3.14.** Show that  $f : \mathbb{H} \rightarrow \mathbb{C}$  corresponds to a section of  $\mathcal{L}^{\otimes k}$  if and only if

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau\right) = (c\tau + d)^k f(\tau).$$

Such a function is called a *modular form of weight  $k$*  for  $SL_2(\mathbb{Z})$ .

**Exercise 3.15.** Show that  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a modular form of weight  $2k$  if and only if the  $k$ -differential (i.e., section of the  $k$ th power of the canonical bundle)

$$f(\tau)d\tau^{\otimes k}$$

is invariant under  $SL_2(\mathbb{Z})$ . Deduce that the (orbifold) canonical bundle of  $\mathcal{M}_1$  is isomorphic to  $\mathcal{L}^{\otimes 2}$ .

Some basic properties and applications of modular forms are given in Chapter VII of [24].

Just as one can define the Picard group of a complex analytic variety to be the group of isomorphism classes of holomorphic line bundles over it, one can define the Picard group  $\text{Pic}_{\text{orb}} X$  of a holomorphic orbifold  $X$  as the group of isomorphism classes of orbifold line bundles over  $X$ . The details are left as a straightforward exercise. The following result will be proved later in the lectures.

**Theorem 3.16.** *The Picard group of the orbifold  $\mathcal{M}_1$  is cyclic of order 12 and generated by the class of  $\mathcal{L}$ .*

It is easy to see that the class of  $\mathcal{L}$  in  $\text{Pic}_{\text{orb}} \mathcal{M}_1$  as  $\mathcal{L}^{\otimes 12}$  is trivialized by the cusp form

$$\Delta(q) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

of weight 12 which has no zeros or poles in  $\mathbb{H}$ . Also, the non-existence of meromorphic modular forms of weight  $k$  with  $0 < k < 12$  and no zeros or poles in  $\mathbb{H}$  implies that  $\mathcal{L}$  is an element of order 12 in  $\text{Pic}_{\text{orb}} \mathcal{M}_1$ . We will show later that the order of  $\text{Pic}_{\text{orb}} \mathcal{M}_1$  is of order 12, from which it will follow that  $\text{Pic}_{\text{orb}} \mathcal{M}_1$  is isomorphic to  $\mathbb{Z}/12$  and generated by the class of  $\mathcal{L}$ .

## LECTURE 2: TEICHMÜLLER THEORY

We have seen in genus 1 case that  $\mathcal{M}_1$  is the quotient

$$\Gamma_1 \backslash \mathcal{X}_1$$

of a contractible complex manifold  $\mathcal{X}_1 = \mathbb{H}$  by a discrete group  $\Gamma_1 = SL_2(\mathbb{Z})$ . The action of  $\Gamma_1$  on  $\mathcal{X}_1$  is said to be *virtually free* — that is,  $\Gamma_1$  has a finite index subgroup which acts (fixed point) freely on  $\mathcal{X}_1$ .<sup>3</sup> In this section we will generalize this to all  $g \geq 1$  — we will sketch a proof that there is a contractible complex manifold  $\mathcal{X}_g$ , called *Teichmüller space*, and a group  $\Gamma_g$ , called the *mapping class group*,

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<sup>3</sup>More generally, we say that a group  $G$  has a property  $P$  *virtually* if  $P$  holds for some finite index subgroup of  $G$ .

which acts virtually freely on  $\mathcal{X}_g$ . The moduli space of genus  $g$  compact Riemann surfaces is the quotient:

$$\mathcal{M}_g = \Gamma_g \backslash \mathcal{X}_g.$$

This will imply that  $\mathcal{M}_g$  has the structure of a complex analytic variety with finite quotient singularities.

Teichmüller theory is a difficult and technical subject. Because of this, it is only possible to give an overview.

#### 4. THE UNIFORMIZATION THEOREM

Our basic tool is the the generalization of the Riemann Mapping Theorem known as the Uniformization Theorem. You can find a proof, for example, in [9] and [8].

**Theorem 4.1.** *Every simply connected Riemann surface<sup>4</sup> is biholomorphic to either  $\mathbb{P}^1$ ,  $\mathbb{C}$  or  $\mathbb{H}$ .*

**Exercise 4.2.** Show that no two of  $\mathbb{P}^1$ ,  $\mathbb{H}$  and  $\mathbb{C}$  are isomorphic as Riemann surfaces.

We have already seen that  $\text{Aut } \mathbb{P}^1 \cong PSL_2(\mathbb{C})$  and that  $PSL_2(\mathbb{R}) \subseteq \text{Aut } \mathbb{H}$ .

**Exercise 4.3.** (i) Show that

$$\text{Aut } \mathbb{C} = \{z \mapsto az + b : a \in \mathbb{C}^* \text{ and } b \in \mathbb{C}\}.$$

- (ii) Show that there is an element  $T$  of  $\text{Aut } \mathbb{P}^1$  that restricts to an isomorphism  $T : \Delta \rightarrow \mathbb{H}$  between the unit disk and the upper half plane.
- (iii) Show that every element of  $\text{Aut } \Delta$  is a fractional linear transformation. Hint: Use the Schwartz Lemma.
- (iv) Deduce that every element of  $\text{Aut } \mathbb{H}$  is a fractional linear transformation and that  $\text{Aut } \mathbb{H} \cong PSL_2(\mathbb{R})$ .

If  $X$  is a Riemann surface and  $x \in X$ , then  $\pi_1(X, x)$  acts on the universal covering  $\tilde{X}$  as a group of biholomorphisms. This action is fixed point free.

**Exercise 4.4.** Show that if  $X$  is a Riemann surface whose universal covering is isomorphic to  $\mathbb{H}$  and  $x \in X$ , then there is a natural injective homomorphism  $\rho : \pi_1(X, x) \rightarrow PSL_2(\mathbb{R})$  which is injective and has discrete image. Show that this homomorphism is unique up to conjugation by an element of  $PSL_2(\mathbb{R})$ , and that the conjugacy class of  $\rho$  is

---

<sup>4</sup>We follow the convention that every Riemann surface is, by definition, connected

independent of the choice of  $x$ . Show that  $X$  is isomorphic to  $\text{im } \rho \backslash \mathbb{H}$  and that the conjugacy class of  $\rho$  determines  $X$  up to isomorphism.

This will give a direct method of putting a topology on  $\mathcal{M}_g$  when  $g \geq 2$ . But first some preliminaries.

**Exercise 4.5.** Show that if  $X$  is a Riemann surface whose universal covering is

- (i)  $\mathbb{P}^1$ , then  $X$  is isomorphic to  $\mathbb{P}^1$ ;
- (ii)  $\mathbb{C}$ , then  $X$  is isomorphic to  $\mathbb{C}$ ,  $\mathbb{C}^*$  or is a genus 1 curve.

Hint: Classify the subgroups of  $\text{Aut } \mathbb{P}^1$  and  $\text{Aut } \mathbb{C}$  that act properly discontinuously and freely.

So we come to the striking conclusion that the universal covering of a Riemann surface not isomorphic to  $\mathbb{P}^1$ ,  $\mathbb{C}$ ,  $\mathbb{C}^*$  or a genus 1 curves must be isomorphic to  $\mathbb{H}$ . In particular, the universal covering of  $\mathbb{C} - \{0, 1\}$  is  $\mathbb{H}$ . Picard's Little Theorem is a consequence.

**Exercise 4.6.** Show that if  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a holomorphic mapping that omits 2 distinct points of  $\mathbb{C}$ , then  $f$  is constant.

For our immediate purposes, the most important fact is that the universal covering of every Riemann surface of genus  $g \geq 2$  is isomorphic to  $\mathbb{H}$ . More generally, we have the following interpretation of the stability condition  $2g - 2 + n > 0$ .

**Exercise 4.7.** Show that if  $U$  is a Riemann surface of the form  $X - F$  where  $X$  is a compact Riemann surface genus  $g$  and  $F$  a finite subset of  $X$  of cardinality  $n$ , then the universal covering of  $U$  is isomorphic to  $\mathbb{H}$  if and only if  $2g - 2 + n > 0$ . Observe that  $2 - 2g - n$  is the topological Euler characteristic of  $U$ .

## 5. TEICHMÜLLER SPACE

Suppose that  $g \geq 2$  and that  $S$  is a compact oriented surface of genus  $g \geq 2$ . Fix a base point  $x_o \in S$  and set  $\pi = \pi_1(S, x_o)$ . Let

$$\mathcal{X}_g = \left\{ \begin{array}{l} \text{conjugacy classes of injective representations} \\ \rho : \pi \hookrightarrow PSL_2(\mathbb{R}) \text{ such that } \text{im } \rho \text{ acts freely on} \\ \mathbb{H} \text{ and } \text{im } \rho \backslash \mathbb{H} \text{ is a compact Riemann surface} \\ \text{of genus } g \end{array} \right\}$$

It is standard that we can choose generators  $a_1, \dots, a_g, b_1, \dots, b_g$  of  $\pi$  that are subject to the relation

$$\prod_{j=1}^g [a_j, b_j] = 1.$$



where  $[x, y]$  denotes the commutator  $xyx^{-1}y^{-1}$ .

**Exercise 5.1.** Show that  $\mathcal{X}_g$  can also be identified naturally with the set

$$\left\{ \begin{array}{l} \text{compact Riemann surfaces } C \text{ plus a conjugacy class of iso-} \\ \text{morphisms } \pi \cong \pi_1(C) \text{ modulo the obvious isomorphisms} \end{array} \right\}$$

The group  $PSL_2(\mathbb{R})$  is a real affine algebraic group. It can be realized as a closed subgroup of  $GL_3(\mathbb{R})$  defined by real polynomial equations.

**Exercise 5.2.** Denote the standard 2-dimensional representation of  $SL_2(\mathbb{R})$  by  $V$ . Denote the second symmetric power of  $V$  by  $S^2V$ . Show that this is a 3-dimensional representation of  $SL_2(\mathbb{R})$ . Show that  $-I \in SL_2(\mathbb{R})$  acts trivially on  $S^2V$ , so that  $S^2V$  is a 3-dimensional representation of  $PSL_2(\mathbb{R})$ . Show that this representation is faithful and that its image is defined by polynomial equations. Deduce that  $PSL_2(\mathbb{R})$  is a real affine algebraic group.

Because of this, we will think of elements of  $PSL_2(\mathbb{R})$  as  $3 \times 3$  matrices whose entries satisfy certain polynomial equations. A representation  $\rho : \pi \rightarrow PSL_2(\mathbb{R})$  thus corresponds to a collection of matrices

$$A_1, \dots, A_g, B_1, \dots, B_g$$

in  $PSL_2(\mathbb{R}) \subset GL_3(\mathbb{R})$  that satisfy the polynomial equation

$$(4) \quad I - \prod_{j=1}^g [A_j, B_j] = 0;$$

the correspondence is given by setting  $A_j = \rho(a_j)$  and  $B_j = \rho(b_j)$ .

The set of all representations  $\rho : \pi \rightarrow PSL_2(\mathbb{R})$  is the real algebraic subvariety  $\mathcal{R}$  of  $PSL_2(\mathbb{R})^{2g}$  consisting of all  $2g$ -tuples

$$(A_1, \dots, A_g, B_1, \dots, B_g)$$

that satisfy (4). This is a closed subvariety of  $PSL_2(\mathbb{R})^{2g}$  and is therefore an affine variety. Note that<sup>5</sup>

$$\dim \mathcal{R} \geq 2g \cdot \dim PSL_2(\mathbb{R}) - \dim PSL_2(\mathbb{R}) = 6g - 3.$$

Observe that  $PSL_2(\mathbb{R})$  acts on  $\mathcal{R}$  on the right by conjugation:

$$A : \rho \mapsto \{g \mapsto A^{-1}\rho(g)A\}.$$

---

<sup>5</sup>One has to be more careful here as, in real algebraic geometry, affine varieties of higher codimension can all be cut out by just one equation. Probably the best way to proceed is to compute the derivative of the product of commutators map  $PSL_2(\mathbb{R})^{2g} \rightarrow PSL_2(\mathbb{R})$  at points in the fiber over the identity.

**Exercise 5.3.** Show that

$$U := \left\{ \rho : \pi \hookrightarrow PSL_2(\mathbb{R}) \text{ such that } \text{im } \rho \text{ acts} \right. \\ \left. \begin{array}{l} \text{freely on } \mathbb{H} \text{ and } \text{im } \rho \backslash \mathbb{H} \text{ is a compact} \\ \text{Riemann surface of genus } g \end{array} \right\}$$

is an open subset of  $\mathcal{R}$  and that it is closed under the  $PSL_2(\mathbb{R})$ -action. Hint: One can construct a fundamental domain  $F$  for the action of  $\pi$  on  $\mathbb{H}$  given by  $\rho$  as follows. First choose a point  $x \in \mathbb{H}$ . Then take  $F$  to be all points in  $\mathbb{H}$  that are closer to  $x$  than to any of the points  $\rho(g)x$  (with respect to hyperbolic distance) where  $g \in \pi$  is non-trivial. Then  $F$  is a compact subset of  $\mathbb{H}$  whose boundary is a union of geodesic segments. As  $\rho$  varies continuously, the orbit of  $x$  varies continuously. Now choose  $x$  to be generic enough and study the change in  $F$  as  $\rho$  varies.

Note that  $\mathcal{X}_g = U/PSL_2(\mathbb{R})$ .

**Exercise 5.4.** Show that the center of  $PSL_2(\mathbb{R})$  is trivial.

**Proposition 5.5.** *If  $\rho \in U$ , then the stabilizer in  $PSL_2(\mathbb{R})$  of  $\rho$  is trivial.*

*Proof.* The first step is to show that if  $\rho \in U$ , then  $\text{im } \rho$  is Zariski dense in  $PSL_2(\mathbb{R})$ . To prove this, it suffices to show that  $\text{im } \rho$  is Zariski dense in the set of complex points  $PSL_2(\mathbb{C})$ . From Lie theory (or algebraic group theory), we know that all proper subgroups of  $PSL_2(\mathbb{C})$  are extensions of a finite group by a solvable group. Since  $\text{im } \rho$  is isomorphic to  $\pi$ , and since  $\pi$  is not solvable (as it contains a free group of rank  $g$ ),  $\text{im } \rho$  cannot be contained in any proper algebraic subgroup of  $PSL_2(\mathbb{R})$ , and is therefore Zariski dense.

An element  $A$  of  $PSL_2(\mathbb{R})$  stabilizes  $\rho$  if and only if  $A^{-1}\rho(g)A = \rho(g)$  for all  $g \in \pi$  — that is, if and only if  $A$  centralizes  $\text{im } \rho$ . But  $A$  centralizes  $\text{im } \rho$  if and only if it centralizes the Zariski closure of  $\text{im } \rho$ . Since  $\text{im } \rho$  is Zariski dense in  $PSL_2(\mathbb{R})$ , and since the center of  $PSL_2(\mathbb{R})$  is trivial, it follows that  $A$  is trivial.  $\square$

**Theorem 5.6.** *The set  $U$  is a smooth manifold of real dimension  $6g - 3$ . The group  $PGL_2(\mathbb{R})$  acts principally on  $U$ , so that quotient  $\mathcal{X}_g = U/PSL_2(\mathbb{R})$  is a manifold of real dimension  $6g - 6$ .*

*Sketch of Proof.* It can be shown, using deformation theory, that the Zariski tangent space of  $\mathcal{R}$  at the representation  $\rho$  of  $\pi$  is given by the

relative cohomology group:<sup>6</sup>

$$T_\rho \mathcal{R} = H^1(S, x_o; \mathbb{A})$$

where  $\mathbb{A}$  is the local system (i.e., locally constant sheaf) over  $S$  whose fiber over  $x_o$  is  $\mathfrak{sl}_1(\mathbb{R})$  and whose monodromy representation is the homomorphism

$$\pi \xrightarrow{\rho} PSL_2(\mathbb{R}) \xrightarrow{\text{Ad}} \text{Aut } \mathfrak{sl}_2(\mathbb{R}).$$

Here  $\mathfrak{sl}_2(\mathbb{R})$  denotes the Lie algebra of  $PSL_2(\mathbb{R})$ , which is the set of  $2 \times 2$  matrices of trace 0, and  $\text{Ad}$  is the adjoint representation

$$A \mapsto \{X \mapsto AXA^{-1}\}.$$

A result equivalent to this was first proved by André Weil [26]. The long exact sequence of the pair  $(S, x_o)$  with coefficients in  $\mathbb{A}$  gives a short exact sequence

$$0 \rightarrow \mathfrak{sl}_2(\mathbb{R}) \rightarrow H^1(S, x_o; \mathbb{A}) \rightarrow H^1(S; \mathbb{A}) \rightarrow 0$$

and an isomorphism  $H^2(S, x_o; \mathbb{A}) \cong H^2(S; \mathbb{A})$ .

Deformation theory also tells us that if  $H^2(S, x_o; \mathbb{A})$  vanishes, then  $\mathcal{R}$  is smooth at  $[\rho] \in \mathcal{R}$ .<sup>7</sup> The Killing form is the symmetric bilinear form on  $\mathfrak{sl}_2(\mathbb{R})$  given by

$$b(X, Y) = \text{tr}(XY).$$

It is non-degenerate and invariant under  $\text{Ad} \circ \rho$ . It follows that  $\mathbb{A}$  is isomorphic to its dual  $\mathbb{A}^*$  as a local system. Thus it follows from Poincaré duality (with twisted coefficients) that

$$H^2(S, \mathbb{A}) \cong H^0(S, \mathbb{A})^*.$$

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<sup>6</sup>This is a non-standard way of giving this group — Weil used cocycles. Here  $A^\bullet(S; \mathbb{A})$  is the complex of differential forms on  $S$  with coefficients in the local system  $\mathbb{A}$ . It is a differential graded Lie algebra. Restricting to the base point  $x_o$  one obtains an augmentation  $A^\bullet(S; \mathbb{A}) \rightarrow \mathfrak{sl}_2(\mathbb{R})$ . The complex  $A^\bullet(S, x_o; \mathbb{A})$  is the kernel of this, and  $H^\bullet(S, x_o; \mathbb{A})$  is defined to be the cohomology of this complex. Briefly, this is related to deformations of  $\rho$  as follows: when the representation  $\rho$  is deformed, the vector bundle underlying the local system corresponding to  $\rho$  does not change, only the connection on it. But a new connection on the bundle corresponding to  $\rho$  will differ from the original connection by an element of  $A^1(S; \mathbb{A})$ . So a deformation of  $\rho$  corresponds to a family of sections  $w(t)$  of  $A^1(S; \mathbb{A})$  where  $w(0) = 0$ . For each  $t$  the new connection is flat. This will imply that the  $t$  derivative of  $w(t)$  at  $t = 0$  is closed in  $A^\bullet(S, \mathbb{A})$ . Changes of gauge that do not alter the marking of the fiber over  $x_o$  are, to first order, elements of  $A^0(S, x_o; \mathbb{A})$ . The  $t$  derivative of such first order changes of gauge gives the trivial first order deformations of  $\rho$ .

<sup>7</sup>A proof of this and also a proof of Weil's result can be found in the beautiful paper of Goldman and Millson [10].

But if  $\rho$  is an element of  $U$ , it follows from Proposition 5.5 that  $H^0(S, \mathbb{A})$  vanishes. Consequently, if  $\rho \in U$ , then

$$\dim H^1(S, \mathbb{A}) = -\chi(S, \mathbb{A}) = -(\text{rank } \mathbb{A}) \cdot \chi(S) = 3(2g - 2) = 6g - 6$$

and

$$\dim H^1(S, x_o; \mathbb{A}) = 6g - 3.$$

We saw above that  $U$  has dimension  $\geq 6g - 3$ , but  $\dim U$  is not bigger than the dimension of any of its Zariski tangent spaces, which we have just shown is  $6g - 3$ . It follows that  $\dim U = 6g - 3$  and that it is smooth as its dimension equals the rank of its Zariski tangent space. Finally, since the  $PSL_2(\mathbb{R})$ -action on  $U$  is principal,  $\mathcal{X}_g = U/PSL_2(\mathbb{R})$  is smooth of dimension  $(6g - 3) - \dim PSL_2(\mathbb{R}) = 6g - 6$ .  $\square$

It is not very difficult to show that the tangent space of  $\mathcal{X}_g$  at  $[\rho]$  is naturally isomorphic to  $H^1(S, \mathbb{A})$ .

## 6. MAPPING CLASS GROUPS

The mapping class group  $\Gamma_g$  plays a role for higher genus ( $g \geq 2$ ) Riemann surfaces analogous to that played by  $SL_2(\mathbb{Z})$  in the theory of genus 1 curves. As in the previous section,  $S$  is a fixed compact oriented surface of genus  $g$ ,  $x_o \in S$ , and  $\pi = \pi_1(S, x_o)$ . But unlike the last section, we allow any  $g \geq 1$ .

The mapping class group  $\Gamma_S$  of  $S$  is defined to be the group of connected components of the group of orientation preserving diffeomorphisms of  $S$ :

$$\Gamma_S = \pi_0 \text{Diff}^+ S.$$

Here  $\text{Diff } S$  is given the compact open topology. It is the finest topology on  $\text{Diff } S$  such that a function  $f : K \rightarrow \text{Diff } S$  from a compact space into  $\text{Diff } S$  is continuous if and only if the map  $\phi_f : K \times S \rightarrow S$  defined by

$$\phi_f(k, x) = f(k)(x)$$

is continuous.

**Exercise 6.1.** Suppose that  $M$  is a smooth manifold. Show that a path in  $\text{Diff } M$  from  $\phi$  to  $\psi$  is a homotopy  $\Psi : [0, 1] \times M \rightarrow M$  from  $\phi$  to  $\psi$  such that for each  $t \in [0, 1]$ , the function  $\Psi(t, \cdot) : M \rightarrow M$  that takes  $x$  to  $\Psi(t, x)$  is a diffeomorphism. Such a homotopy is called an *isotopy* between  $\phi$  and  $\psi$ .

Recall that the outer automorphism group of a group  $G$  is defined to be the quotient

$$\text{Out } G = \text{Aut } G / \text{Inn } G$$

of the automorphism group  $\text{Aut } G$  of  $G$  by the subgroup  $\text{Inn } G$  of inner automorphisms.

**Exercise 6.2.** Suppose that  $M$  is a smooth manifold. Show that each diffeomorphism of  $M$  induces an outer automorphism of its fundamental group and that the corresponding function

$$\text{Diff } M \rightarrow \text{Out } \pi(M)$$

is a homomorphism. (Hint: Fix a base point of  $M$  and show that every diffeomorphism of  $M$  is isotopic to one that fixes the base point.) Show that the kernel of this homomorphism contains the subgroup of all diffeomorphisms isotopic to the identity. Deduce that there is a homomorphism

$$\pi_0(\text{Diff } M) \rightarrow \text{Out } \pi_1(M).$$

Taking  $M$  to be  $S$ , our reference surface, we see that there is a natural homomorphism

$$\Phi : \Gamma_S \rightarrow \text{Out } \pi.$$

It is a remarkable fact, due to Dehn [6] and Nielsen [22], that this map is almost an isomorphism. (I believe there is a proof of this in the translation of some papers of Dehn on topology by Stillwell.)

**Theorem 6.3** (Dehn-Nielsen). *For all  $g \geq 1$ , the homomorphism  $\Phi$  is injective and the image of  $\Phi$  is a subgroup of index 2.*  $\square$

One can describe the index 2 subgroup of  $\text{Out } \pi$  as follows. Since  $S$  has genus  $\geq 1$ , the universal covering of  $S$  is contractible, and there is a natural isomorphism  $H_i(S, \mathbb{Z}) \cong H_i(\pi, \mathbb{Z})$ . Each element of  $\text{Out } \pi$  induces an automorphism of  $H_i(\pi, \mathbb{Z})$  as inner automorphisms of a group act trivially on its cohomology. The image of  $\Phi$  is the kernel of the natural homomorphism

$$\text{Out } \pi \rightarrow \text{Aut } H_2(\pi, \mathbb{Z}) \cong \{\pm 1\}.$$

When  $g = 1$ ,  $\pi \cong \mathbb{Z}^2$ , and

$$\text{Out } \pi = \text{Aut } \pi \cong GL_2(\mathbb{Z}).$$

**Exercise 6.4.** Show that when  $g = 1$ , the homomorphism  $\text{Out } \pi \rightarrow \{\pm 1\}$  corresponds to the determinant  $GL_2(\mathbb{Z}) \rightarrow \{\pm 1\}$ .

The genus 1 case of the Dehn-Nielsen Theorem can be proved by elementary means:

**Exercise 6.5.** Show that if  $g = 1$ , then the homomorphism  $\Gamma_S \rightarrow SL_2(\mathbb{Z})$  is an isomorphism. Hint: Construct a homomorphism from  $SL_2(\mathbb{Z})$  to  $\text{Diff } S$ .

Since two compact orientable surfaces are diffeomorphic if and only if they have the same genus, it follows that the group  $\Gamma_S$  depends only on the genus of  $S$ . For this reason, we define  $\Gamma_g$  to be  $\Gamma_S$  where  $S$  is any compact genus  $g$  surface.

## 7. THE MODULI SPACE

In this section,  $S$  is once again a compact oriented surface of genus  $g \geq 2$  and  $\pi$  denotes its fundamental group. The mapping class group  $\Gamma_g = \Gamma_S$  acts smoothly on Teichmüller space  $\mathcal{X}_g$  on the left via the homomorphism  $\Gamma_g \rightarrow \text{Out } \pi$ .

**Exercise 7.1.** Describe this action explicitly. Show that the isotropy group of  $[\rho] \in \mathcal{X}_g$  is naturally isomorphic to the automorphism group of the Riemann surface  $\text{im } \rho \backslash \mathbb{H}$ . (Compare this with the genus 1 case.) Deduce that all isotropy groups are finite.

**Exercise 7.2.** Show that two points in  $\mathcal{X}_g$  are in the same  $\Gamma_g$  orbit if and only if they determine the same Riemann surface. Hint: Use the Uniformization Theorem.

Rephrasing this, we get:

**Theorem 7.3.** *If  $g \geq 2$ , then  $\mathcal{M}_g$  is naturally isomorphic to the quotient of  $\mathcal{X}_g$  by  $\Gamma_g$ .*  $\square$

This result allows us to put a topology on  $\mathcal{M}_g$ . We give it the unique topology so that  $\mathcal{X}_g \rightarrow \mathcal{M}_g$  is a quotient mapping. To try to understand the topology of  $\mathcal{X}_g$  we shall need the following fundamental result.

**Theorem 7.4** (Teichmüller). *For all  $g \geq 2$ , the Teichmüller space  $\mathcal{X}_g$  is contractible and the action of  $\Gamma_g$  on it is properly discontinuous.*

A natural way to approach the proof of this theorem via hyperbolic geometry. We do this in the next two sections.

## 8. HYPERBOLIC GEOMETRY

Perhaps the most direct way to approach the study of Teichmüller space is via hyperbolic geometry. The link between complex analysis and hyperbolic geometry comes from the fact that the upper half plane has a complete metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

of constant curvature  $-1$  whose group of orientation preserving isometries is  $PSL_2(\mathbb{R})$ . The Riemannian manifold  $(\mathbb{H}, ds^2)$  is called the

*Poincaré upper half plane.* A good reference for hyperbolic geometry is the book by Beardon [2].

**Exercise 8.1.** Show that

- (i) every element of  $PSL_2(\mathbb{R})$  is an orientation preserving isometry of  $(\mathbb{H}, ds^2)$ ;
- (ii)  $PSL_2(\mathbb{R})$  acts transitively on  $\mathbb{H}$  and that the stabilizer of one point (say  $i$ ), and therefore all points, of  $\mathbb{H}$  is isomorphic to  $SO(2)$  and acts transitively on the unit tangent space of  $\mathbb{H}$  at this point.

Deduce that the Poincaré metric  $ds^2$  has constant curvature and that  $PSL_2(\mathbb{R})$  is the group of all orientation preserving isometries of the Poincaré upper half plane.

The fact that the group of biholomorphisms of  $\mathbb{H}$  coincides with the group of orientation preserving isometries of  $\mathbb{H}$  is fundamental and is used as follows. Suppose that  $X$  is a Riemann surface whose universal covering is  $\mathbb{H}$ . Then, by Exercise 4.4,  $X$  is biholomorphic to the quotient of  $\mathbb{H}$  by a discrete subgroup  $\Gamma$  of  $PSL_2(\mathbb{R})$  isomorphic to  $\pi_1(X)$ . Now, since  $PSL_2(\mathbb{R})$  is also the group of orientation preserving isometries of  $\mathbb{H}$ , the Poincaré metric  $ds^2$  is invariant under  $\Gamma$  and therefore descends to  $X = \Gamma \backslash \mathbb{H}$ . This metric is complete as the Poincaré metric is.

**Exercise 8.2.** Prove the converse of this: if  $X$  is an oriented surface with a complete hyperbolic metric, then  $X$  has a natural complex structure whose canonical orientation agrees with the original orientation.

These two constructions are mutually inverse. Thus we conclude that when  $g \geq 2$ , there is a natural one-to-one correspondence

$$(5) \quad \left\{ \begin{array}{l} \text{isomorphism classes of compact} \\ \text{Riemann surfaces of genus } g \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{isometry classes of compact oriented sur-} \\ \text{faces of genus } g \text{ with a hyperbolic metric} \end{array} \right\}$$

It follows that if  $g \geq 2$ , then

$$\mathcal{M}_g = \left\{ \begin{array}{l} \text{isometry classes of compact oriented sur-} \\ \text{faces of genus } g \text{ with a hyperbolic metric} \end{array} \right\}.$$

*Remark 8.3.* Likewise, a Riemann surface of genus 1 has a flat metric (unique up to rescaling) which determines the complex structure. So  $\mathcal{M}_1$  can be regarded as the moduli space of (conformal classes of) flat tori.

**Exercise 8.4.** Show that if  $X$  is a Riemann surface whose universal covering is  $\mathbb{H}$ , then the group of orientation preserving isometries of  $X$  equals the group of biholomorphisms of  $X$ .

## 9. FENCHEL-NIELSEN COORDINATES

Once again, we assume that  $g \geq 2$  and that  $S$  is a reference surface of genus  $g$ . The interpretation of Teichmüller space in terms of hyperbolic geometry allows us to define coordinates on  $\mathcal{X}_g$ . To do this we decompose the surface into “pants.”

A *pair of pants* is a compact oriented surface of genus 0 with 3 boundary components. Alternatively, a pair of pants is a disk with 2 holes. A *simple closed curve* (SCC) on  $S$  is an imbedded circle. A

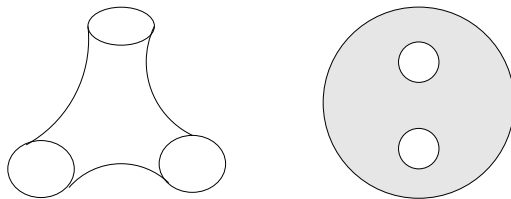


FIGURE 2. pairs of pants

A *pants decomposition* of  $S$  is a set of disjoint simple closed curves in  $S$  that divides  $S$  into pairs of pants.

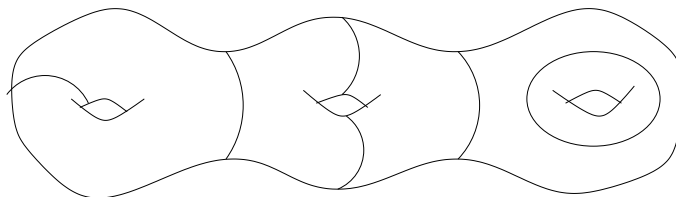


FIGURE 3. a pants decomposition

**Proposition 9.1.** *If the simple closed curves  $C_1, \dots, C_n$  divide  $S$  into  $m$  pairs of pants, then  $n = 3g - 3$  and  $m = 2g - 2$ .*

*Proof.* Since a pair of pants has the homotopy type of a bouquet of 2 circles, it has Euler characteristic  $-1$ . Since the Euler characteristic of a disjoint union of circles is 0, we have

$$2 - 2g = \chi(S) = m \cdot \chi(\text{a pair of pants}) = -m.$$

Thus  $m = 2g - 2$ . Since each pair of pants has 3 boundary components, and since each circle lies on the boundary of two pairs of pants, we see



that

$$n = 3m/2 = 3g - 3.$$

□

If  $T$  is a compact oriented surface with boundary and  $\chi(T) < 0$ , then  $T$  has a hyperbolic metric such that each boundary component is totally geodesic. As in the case where the boundary of  $T$  is empty, the set of all hyperbolic structures on  $T$  (modulo diffeomorphisms isotopic to the identity) can be described as a representation variety. I will omit the details. This space is called the Teichmüller space of  $T$  and will be denoted by  $\mathcal{X}_T$ .

**Proposition 9.2.** *If  $P$  is a pair of pants, then  $\mathcal{X}_P$  is a manifold diffeomorphic to  $\mathbb{R}^3$ . The diffeomorphism is given by taking a hyperbolic structure on  $P$  to the lengths of its 3 boundary components.*

*Discussion of the Proof.* A proof can be found in the Exposé 3, partie II by Poénaru in the book [7]. The basic idea is that a pair of pants can be cut into two isomorphic hyperbolic right hexagons, and that two right hyperbolic hexagons are equivalent if the lengths of every other side of one equal the lengths of the corresponding sides of the other. Also, one can construct hyperbolic right hexagons where these lengths are arbitrary positive real numbers. □

A basic fact in hyperbolic geometry is that if one has a compact hyperbolic surface  $T$  (not assumed to be connected) with totally geodesic boundary, then one can identify pairs of boundary components of the same lengths to obtain a hyperbolic surface whose new hyperbolic structure agrees with that on  $T - \partial T$ . Another elementary fact we shall need is that if  $C$  is a SCC in a compact hyperbolic surface  $T$  with totally geodesic boundary, then there is a unique closed geodesic  $\gamma : S^1 \rightarrow T$  that is freely homotopic to  $C$ .

Now suppose that  $\mathcal{P} = \{P_1, \dots, P_{2g-2}\}$  is a pants decomposition of  $S$  determined by the SCCs  $C_1, \dots, C_{3g-3}$ . For each hyperbolic structure on  $S$ , we can find closed geodesics  $\gamma_1, \dots, \gamma_{3g-3}$  that are isotopic to the  $C_j$ . Taking the lengths of these, we obtain a function

$$\ell : \mathcal{X}_S \rightarrow \mathbb{R}_+^{3g-3}.$$

It is not difficult to show that this mapping is continuous. (See [7], for example.) On the other hand, we can define an action of  $\mathbb{R}^{3g-3}$  on  $\mathcal{X}_S$  and whose orbits lie in the fibers of  $\ell$ . To define the action of  $(\theta_1, \dots, \theta_{3g-3})$  on a hyperbolic surface  $S$ , write  $S$  as the union of  $2g - 2$  pairs of hyperbolic pants whose boundaries are represented by

geodesics  $\gamma_1, \dots, \gamma_{3g-3}$  isotopic to the  $C_j$ . Then twist the gluing map at  $\gamma_j$  by an angle  $\theta_j$ . This is also continuous.

**Theorem 9.3** (Douady, [7, Exposé 7]). *If  $g \geq 2$ , then the map  $\ell : \mathcal{X}_g \rightarrow \mathbb{R}_+^{3g-3}$  is a principal  $\mathbb{R}^{3g-3}$  bundle with the action described above.*  $\square$

**Corollary 9.4.** *For all  $g \geq 2$ , Teichmüller space  $\mathcal{X}_g$  is diffeomorphic to  $\mathbb{R}^{6g-6}$ .*  $\square$

**Exercise 9.5.** At first it may appear that  $\ell : \mathcal{X}_g \rightarrow \mathbb{R}_+^{3g-3}$  should be a principal  $(S^1)^{3g-3}$  bundle. Show that rotation by  $2\pi$  about one of the SCCs  $C_j$  alters the representation  $\rho$  by an automorphism of  $\pi$  that is not inner. Hint: it may help to first read the part of Section 15 on Dehn twists.

## 10. THE COMPLEX STRUCTURE

Suppose that  $C$  is a compact Riemann surface. Recall that a deformation of  $C$  is the germ about  $t_o \in T$  of a proper analytic mapping  $f : \mathcal{C} \rightarrow T$ , where  $T$  is an analytic variety, and an isomorphism  $j : C \rightarrow f^{-1}(t_o)$ . One defines maps between deformations to be cartesian squares. A deformation of  $C$  is called universal if every other deformation is pulled back from it. A standard result in deformation theory is that every Riemann surface of genus  $g \geq 2$  has a universal deformation. This has the property that  $T$  is smooth at  $t_o$  and that  $T$  is of complex dimension  $3g - 3$  with tangent space at  $t_o$  canonically isomorphic to  $H^1(C, \Theta_C)$ , where  $\Theta_C$  denotes the sheaf of holomorphic sections of the tangent bundle of  $C$ .

As explained in Looijenga's lectures,  $\mathcal{M}_g$  can be obtained by patching such local deformation spaces together. If  $C$  is a smooth projective curve of genus  $g$  and  $\rho : \pi \rightarrow PSL_2(\mathbb{R})$  is a representation such that  $C \cong \text{im } \rho \backslash \mathbb{H}$ , then  $[\rho] \in \mathcal{X}_g$  goes to  $[C] \in \mathcal{M}_g$  under the projection  $\mathcal{X}_g \rightarrow \mathcal{M}_g$ . The following fact can be proved using partial differential equations.

**Theorem 10.1.** *If  $(T, t_o)$  is a universal deformation space for  $C$ , then there is a smooth mapping  $\psi : T \rightarrow \mathcal{X}_g$  such that  $\psi(t_o) = [\rho]$  which is a diffeomorphism in a neighbourhood of  $t_o$ . Moreover, the composition of  $\psi$  with the projection  $\mathcal{X}_g \rightarrow \mathcal{M}_g$  is the canonical mapping that classifies the universal deformation of  $C$ .*  $\square$

**Corollary 10.2.** *There is a canonical  $\Gamma_g$  invariant complex structure on  $\mathcal{X}_g$  such that the projection  $\mathcal{X}_g \rightarrow \mathcal{M}_g$  is a complex analytic mapping.*  $\square$

It should be noted that, although  $\mathcal{X}_g$  is a complex manifold diffeomorphic to  $\mathbb{R}^{6g-6}$ , it is not biholomorphic to either a complex  $3g-3$  ball or  $\mathbb{C}^{3g-3}$  when  $g > 1$ .

Combining these results, we obtain the following basic result:

**Theorem 10.3.** *The moduli space  $\mathcal{M}_g$  is a complex analytic variety whose singularities are all finite quotient singularities. Furthermore,  $\mathcal{M}_g$  can be regarded as an orbifold whose universal covering is the complex manifold  $\mathcal{X}_g$  and whose orbifold fundamental groups can be identified, up to conjugacy, with  $\Gamma_g$ . More precisely, if  $C$  is a compact genus  $g$  curve with no non-trivial automorphisms, then there is a canonical isomorphism*

$$\pi_1^{\text{orb}}(\mathcal{M}_g, [C]) \cong \pi_0 \text{Diff}^+ C. \quad \square$$

## 11. THE TEICHMÜLLER SPACE $\mathcal{X}_{g,n}$

It is natural to expect that there is a complex manifold  $\mathcal{X}_{g,n}$  and a discrete group  $\Gamma_{g,n}$  that acts holomorphically on  $\mathcal{X}_{g,n}$  in such a way that  $\mathcal{M}_{g,n}$  is isomorphic to the quotient  $\Gamma_{g,n} \backslash \mathcal{X}_{g,n}$ . We give a brief sketch of how this may be deduced from the results when  $n = 0$  and  $g \geq 2$ .

First, we fix an  $n$ -pointed reference surface of genus  $g$  where  $g \geq 2$ . That is, we fix a compact oriented surface  $S$  and a subset  $P = \{x_1, \dots, x_n\}$  of  $n$  distinct points of  $S$ . Define

$$\Gamma_{g,n} := \pi_0 \text{Diff}^+(S, P)$$

By definition, elements of  $\text{Diff}^+(S, P)$  are orientation preserving and act trivially on  $P$ .

Here is a sketch of a construction of  $\mathcal{X}_{g,1}$ . One can construct the  $\mathcal{X}_{g,n}$  when  $n > 1$  in a similar fashion.

There is a universal curve  $\mathcal{C} \rightarrow \mathcal{X}_g$ . This can be constructed using deformation theory. This is a fiber bundle in the topological sense and it is not difficult to see that the action of the mapping class group  $\Gamma_g$  on  $\mathcal{X}_g$  can be lifted to an action on  $\mathcal{C}$  such that the projection is equivariant. Since the fiber of  $\mathcal{C} \rightarrow \mathcal{X}_g$  is a compact surface of genus  $g$ , and since  $\mathcal{X}_g$  is contractible,  $\mathcal{C}$  has the homotopy type of a surface of genus  $g$  and therefore has fundamental group isomorphic to  $\pi_1(S)$ . Define  $\mathcal{X}_{g,1}$  to be the universal covering of  $\mathcal{C}$ . This is a complex manifold as  $\mathcal{C}$  is. The fibers of the projection  $\mathcal{X}_{g,1}$  are all isomorphic to  $\mathbb{H}$ . Since  $\mathcal{X}_g$  is contractible, so is  $\mathcal{X}_{g,1}$ .

**Exercise 11.1.** Show that there is a natural bijection

$$\mathcal{X}_{g,1} = \left\{ \begin{array}{l} \text{conjugacy classes of representations } \rho : \pi \rightarrow \\ PSL_2(\mathbb{R}) \text{ such that } C_\rho := \text{im } \rho \backslash \mathbb{H} \text{ is of genus } \\ g, \text{ plus a point } x \in C_\rho \end{array} \right\}.$$

Use this (or otherwise) to show that  $\Gamma_{g,1}$  acts on  $\mathcal{X}_{g,1}$  and that the quotient is  $\mathcal{M}_{g,1}$ . Show that the isotropy group of any point of  $\mathcal{X}_{g,n}$  lying above  $[C; x] \in \mathcal{M}_{g,n}$  is naturally isomorphic to  $\text{Aut}(C, x)$ .

We shall regard  $\mathcal{M}_{g,n}$  as the orbifold  $\Gamma_{g,n} \backslash \mathcal{X}_{g,n}$ . There is a natural isomorphism

$$\pi_1^{\text{orb}}(\mathcal{M}_{g,n}, [C; x_1, \dots, x_n]) \cong \pi_0 \text{Diff}^+(C, \{x_1, \dots, x_n\})$$

provided  $\text{Aut}(C; x_1, \dots, x_n)$  is trivial.

## 12. LEVEL STRUCTURES

Level structures are useful technical devices for rigidifying curves. Suppose that  $\ell$  is a positive integer. A *level  $\ell$  structure* on a compact Riemann surface is the choice of a symplectic basis of  $H_1(C, \mathbb{Z}/\ell\mathbb{Z})$ .

**Exercise 12.1.** Show that if  $C$  is a compact Riemann surface of genus  $g$ , then there is a canonical isomorphism between  $H_1(C, \mathbb{Z}/\ell\mathbb{Z})$  and the  $\ell$ -torsion points in  $\text{Pic}^0 C$ . (Remark: the intersection pairing on  $H_1(C, \mathbb{Z}/\ell\mathbb{Z})$  corresponds to the Weil pairing on the  $\ell$ -torsion points.)

Denote the moduli space of  $n$ -pointed, smooth, genus  $g$  curves with a level  $\ell$  structure by  $\mathcal{M}_{g,n}[\ell]$ . This can be described as a quotient of Teichmüller space by a subgroup of  $\Gamma_{g,n}$  that we now describe.

Fix an  $n$ -pointed compact oriented reference surface  $S$  of genus  $g$ . The mapping class group  $\Gamma_{g,n}$  acts naturally on  $H_1(S, \mathbb{Z})$ . Since it preserves the intersection pairing, this leads to a homomorphism

$$\rho : \Gamma_{g,n} \rightarrow \text{Aut}(H_1(S, \mathbb{Z}), \text{intersection form}).$$

Define the level  $\ell$  subgroup of  $\Gamma_{g,n}$  to be the kernel of the homomorphism

$$\rho_\ell : \Gamma_{g,n} \rightarrow \text{Aut}(H_1(S, \mathbb{Z}/\ell\mathbb{Z}), \text{intersection form}).$$

This homomorphism is surjective.

**Exercise 12.2.** Show that  $\Gamma_{g,n}[\ell]$  is a normal subgroup of finite index in  $\Gamma_{g,n}$  and that the quotient is isomorphic to  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ .

**Exercise 12.3.** Show that there is a natural bijection between  $\mathcal{M}_{g,n}[\ell]$  and the quotient of  $\mathcal{X}_{g,n}$  by  $\Gamma_{g,n}[\ell]$ . Show that the quotient mapping  $\mathcal{M}_{g,n}[\ell] \rightarrow \mathcal{M}_{g,n}$  that forgets the level structure has finite degree and is Galois with Galois group  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ .

Choosing a symplectic basis of  $H_1(S, \mathbb{Z})$  gives an isomorphism

$$\text{Aut}(H_1(S, \mathbb{Z}), \text{intersection form}) \cong Sp_g(\mathbb{Z}).$$

We therefore have a homomorphism

$$\rho : \Gamma_{g,n} \rightarrow Sp_g(\mathbb{Z}).$$

Define the level  $\ell$  subgroup  $Sp_g(\mathbb{Z})[\ell]$  of  $Sp_g(\mathbb{Z})$  to be the kernel of the reduction mod  $\ell$  homomorphism

$$Sp_g(\mathbb{Z}) \rightarrow Sp_g(\mathbb{Z}/\ell\mathbb{Z}).$$

This homomorphism is surjective.

**Exercise 12.4.** (i) Show that  $\Gamma_{g,n}[\ell]$  is the inverse image of  $Sp_g(\mathbb{Z})[\ell]$  under  $\rho$ .  
(ii) Show that  $Sp_1(\mathbb{Z})$  is isomorphic to  $SL_2(\mathbb{Z})$  and that  $\rho$  is the standard representation when  $g = 1$ .

**Theorem 12.5** (Minkowski). *The group  $Sp_g(\mathbb{Z})[\ell]$  is torsion free when  $\ell \geq 3$ .*  $\square$

**Proposition 12.6.** *If  $2g - 2 + n > 0$ ,  $g \geq 1$  and  $\ell \geq 3$ , then the mapping class group  $\Gamma_{g,n}[\ell]$  is torsion free and acts fixed point freely on  $\mathcal{X}_{g,n}$ .*

*Sketch of Proof.* The case  $g = 1$  is left as an exercise. Suppose that  $g \geq 2$ . We first show that  $\Gamma_{g,n}[\ell]$  acts fixed point freely on  $\mathcal{X}_{g,n}$ . The isotropy group of a point in  $\mathcal{X}_{g,n}$  lying over  $[C; x_1, \dots, x_n]$  is isomorphic to  $\text{Aut}(C; x_1, \dots, x_n)$ . This is a finite group, and is a subgroup of  $\text{Aut } C$ . It is standard that the natural representation  $\text{Aut } C \rightarrow \text{Aut } H^0(C, \Omega_C^1)$  is injective (Exercise: prove this. Hint: use Riemann-Roch). It follows that the natural representation

$$\text{Aut } C \rightarrow \text{Aut}(H_1(C, \mathbb{Z}), \text{intersection pairing}) \cong Sp_g(\mathbb{Z}).$$

is injective and that  $\text{Aut}(C; x_1, \dots, x_n) \cap \Gamma_{g,n}[\ell]$  is trivial. (Here we are realizing  $\text{Aut}(C; x_1, \dots, x_n)$  as a subgroup of  $\Gamma_{g,n}$  as an isotropy group.) It follows from Minkowski's Theorem that if  $\ell \geq 3$ , then  $\Gamma_{g,n}[\ell]$  acts fixed point freely on  $\mathcal{X}_{g,n}$ .

The rest of the proof is standard topology. If  $\Gamma_{g,n}[\ell]$  has a torsion element, then it contains a subgroup  $G$  of prime order,  $p$  say. Since this acts fixed point freely on the contractible space  $\mathcal{X}_{g,n}$ , it follows that the quotient  $G \backslash \mathcal{X}_{g,n}$  is a model of the classifying space  $B(\mathbb{Z}/p\mathbb{Z})$  of the cyclic group of order  $p$ . Since the model is a manifold of real dimension  $6g - 6 + 2n$ , this implies that  $H^k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$  vanishes when  $k > 6g - 6 + 2n$ . But this contradicts the known computation that  $H^k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$  is non-trivial for all  $k \geq 0$ . The result follows.  $\square$

Putting together the results of this section, we have:

**Corollary 12.7.** *If  $g \geq 1$ ,  $n \geq 0$  and  $\ell \geq 3$ , then  $\mathcal{M}_{g,n}[\ell]$  is smooth and the mapping  $\mathcal{X}_{g,n} \rightarrow \mathcal{M}_{g,n}[\ell]$  is unramified with Galois group  $\Gamma_{g,n}$ ; the covering  $\mathcal{M}_{g,n}[\ell] \rightarrow \mathcal{M}_{g,n}$  is a finite (ramified) Galois covering with Galois group  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ .  $\square$*

With this information, we are able to prove a non-trivial result about the mapping class group.

**Corollary 12.8** (McCool, Hatcher-Thurston). *For all  $g$  and  $n$ , the mapping class group is finitely presented.*

*Proof.* We shall use the fact that each  $\mathcal{M}_{g,n}[\ell]$  is a quasi-projective variety. As we have seen, this is smooth when  $\ell \geq 3$  and has fundamental group isomorphic to  $\Gamma_{g,n}[\ell]$ . But a well known result Lojasiewicz [19] (see also [18]) implies that every smooth quasi-projective variety has the homotopy type of a finite complex. It follows that  $\Gamma_{g,n}[\ell]$  is finitely presented when  $\ell \geq 3$ . But since  $\Gamma_{g,n}[\ell]$  has finite index in  $\Gamma_{g,n}$ , this implies that  $\Gamma_{g,n}$  is also finitely presented.  $\square$

### 13. COHOMOLOGY

One way to define the homology and cohomology of a group  $G$  is to find a topological space  $X$  such that

$$\pi_j(X, *) = \begin{cases} G & j = 1; \\ 0 & j \neq 1. \end{cases}$$

This occurs, for example, when the universal covering of  $X$  is contractible. Such a space is called a *classifying space of  $G$* . Under some mild hypotheses, it is unique up to homotopy. One then defines the cohomology of  $G$  with coefficients in the  $G$ -module  $V$  by

$$H^j(G, V) = H^j(X, \mathbb{V})$$

where  $\mathbb{V}$  is the locally constant sheaf over  $X$  whose fiber is  $V$  and whose monodromy is given by the action of  $G$  on  $V$ . It is well defined. Homology is defined similarly in terms of the homology of  $X$  with coefficients in  $\mathbb{V}$ .

**Exercise 13.1.** Suppose that  $V$  is a  $\Gamma_{g,n}$ -module and  $\mathbb{V}$  is the corresponding locally constant sheaf over  $\mathcal{M}_{g,n}[\ell]$ , where  $\ell \geq 3$ . Show that there is a natural isomorphism

$$H^\bullet(\mathcal{M}_{g,n}[\ell], \mathbb{V}) \cong H^\bullet(\Gamma_{g,n}[\ell], V).$$

Show that the group  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$  acts on both sides and that the isomorphism is  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ -equivariant.

Since  $\Gamma_{g,n}$  does not act fixed point freely on  $\mathcal{X}_{g,n}$ ,  $\mathcal{M}_{g,n}$  is not a classifying space for  $\Gamma_{g,n}$ . Nonetheless, standard topological arguments give

$$H^\bullet(\mathcal{M}_{g,n}, \mathbb{Q}) \cong H^\bullet(\mathcal{M}_{g,n}, \mathbb{Q})^{Sp_g(\mathbb{Z}/\ell\mathbb{Z})}$$

and

$$H^\bullet(\Gamma_{g,n}, \mathbb{Q}) \cong H^\bullet(\Gamma_{g,n}, \mathbb{Q})^{Sp_g(\mathbb{Z}/\ell\mathbb{Z})}.$$

It follows that:

**Theorem 13.2.** *There is a natural isomorphism*

$$H^\bullet(\Gamma_{g,n}, \mathbb{Q}) \cong H^\bullet(\mathcal{M}_{g,n}, \mathbb{Q}). \quad \square$$

If  $V$  is a  $\Gamma_{g,n}$ -module and  $\Gamma_{g,n}$  has fixed points in  $\mathcal{X}_{g,n}$ , we cannot always define a local system  $\mathbb{V}$  over  $\mathcal{M}_{g,n}$  corresponding to  $V$ . However, we can formally define

$$H^\bullet(\mathcal{M}_g, \mathbb{V} \otimes \mathbb{Q}) := H^\bullet(\mathcal{M}_{g,n}[\ell], \mathbb{V} \otimes \mathbb{Q})^{Sp_g(\mathbb{Z}/\ell\mathbb{Z})}.$$

where the superscript  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$  means that we take the  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$  invariant part. This should be regarded as the cohomology of the orbifold  $\mathcal{M}_{g,n}$  with coefficients in the orbifold local system corresponding to  $V$ .

**Exercise 13.3.** Show that this definition is independent of the choice of the level  $\ell \geq 3$ .

Let  $A$  be the locus of curves in  $\mathcal{M}_g$  with non-trivial automorphisms. The goal of the following exercise is to show that  $A$  is an analytic subvariety of  $\mathcal{M}_g$ , each of whose components has codimension  $\geq g - 2$ . Set

$$\mathcal{M}'_g = \mathcal{M}_g - A.$$

**Exercise 13.4.** Suppose that  $X$  is a Riemann surface of genus  $g \geq 2$  and that  $G$  is a finite subgroup of  $\text{Aut } X$ .

(i) Set  $Y = G \backslash X$ . Show that if  $X$  is compact, then

$$g(X) - 1 = d(g(Y) - 1) + \sum_{\mathcal{O}} (d - |\mathcal{O}|)/2$$

where  $\mathcal{O}$  ranges over the orbits of  $G$  acting on  $X$  and where  $d$  is the order of  $G$ .

(ii) Show that if  $g \geq 3$ , then each component of the locus of curves in  $\mathcal{M}_g$  that have automorphisms is a proper subvariety of  $\mathcal{M}_g$ .

(iii) Show that the hyperelliptic locus in  $\mathcal{M}_g$  has codimension  $g - 2$ .

- (iv) Show that codimension of each component of the locus in  $\mathcal{M}_g$  with curves with a non-trivial automorphism is  $\geq g - 2$ , with equality if and only if the component is the hyperelliptic locus. Hint: reduce to the case where  $d$  is prime.
- (v) Give an argument that there are only finitely many components of the locus in  $\mathcal{M}_g$  of curves with a non-trivial automorphism.

Deduce that when  $g \geq 3$ , the set of points of  $\mathcal{M}_g$  corresponding to curves without automorphisms is Zariski dense in  $\mathcal{M}_g$ .

Since each component of  $A$  has real codimension  $\geq 2g - 4$ , it follows from standard topological arguments (transversality) that the inverse image  $\mathcal{X}'_g$  of  $\mathcal{M}_g$  in  $\mathcal{X}_g$  has the property

$$\pi_j(\mathcal{X}'_g) = 0 \text{ if } j < 2g - 5.$$

From this, one can use standard topology to show that if  $g \geq 3$ , and if  $V$  is any  $\Gamma_g$ -module, then there is a natural mapping

$$H^k(\Gamma_g, V) \rightarrow H^k(\mathcal{M}'_g, \mathbb{V})$$

which is an isomorphism when  $k < 2g - 5$  and injective when  $k = 2g - 5$ .<sup>8</sup> In particular, there is a natural isomorphism

$$H^k(\Gamma_g, \mathbb{Z}) \cong H^k(\mathcal{M}'_g, \mathbb{Z})$$

when  $k \leq 2g - 5$ .

There are similar results for  $\mathcal{M}_{g,n}$ , but the codimension of the locus of curves with automorphisms rises with  $n$ . For example, if  $n > 2g + 2$ , then  $\text{Aut}(C; x_1, \dots, x_n)$  is always trivial by the Lefschetz fixed point formula. (Exercise: prove this.)

### LECTURE 3: THE PICARD GROUP

In this lecture, we compute the orbifold Picard group of  $\mathcal{M}_g$  for all  $g \geq 1$ . Recall that an orbifold line bundle over  $\mathcal{M}_g$  is a holomorphic line bundle  $\mathcal{L}$  over Teichmüller space  $\mathcal{X}_g$  together with an action of the mapping class group  $\Gamma_g$  on it such that the projection  $\mathcal{L} \rightarrow \mathcal{X}_g$  is  $\Gamma_g$ -equivariant. An orbifold section of this line bundle is a holomorphic  $\Gamma_g$ -equivariant section  $\mathcal{X}_g \rightarrow \mathcal{L}$  of  $\mathcal{L}$ . This is easily seen to be equivalent to fixing a level  $\ell \geq 3$  and considering holomorphic line bundles over  $\mathcal{M}_g[\ell]$  with an  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ -action such that the projection is  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ -equivariant. Working on  $\mathcal{M}_g[\ell]$  has the advantage that we can talk about algebraic line bundles more easily.

An *algebraic orbifold line bundle* over  $\mathcal{M}_g$  is an algebraic line bundle over  $\mathcal{M}_g[\ell]$  for some  $\ell$  equipped with an action of  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$  such that

---

<sup>8</sup>There is a similar result for homology with the arrows reversed and injectivity replaced by surjectivity.



the projection to  $\mathcal{M}_g[\ell]$  is  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ -equivariant. A section of such a line bundle is simply an  $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ -equivariant section defined over  $\mathcal{M}_g[\ell]$ . Isomorphism of such orbifold line bundles is defined in the obvious way. Let

$$\mathrm{Pic}_{\mathrm{orb}} \mathcal{M}_g$$

denote the group of isomorphism classes of algebraic orbifold line bundles over  $\mathcal{M}_g$ . Our goal in this lecture is to compute this group. It is first useful to review some facts about the Picard group of a smooth projective variety.

#### 14. GENERAL FACTS

Recall that if  $X$  is a compact Kähler manifold (such as a smooth projective variety), then the exponential sequence gives an exact sequence

$$0 \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow \mathrm{Pic} X \rightarrow H^2(X, \mathbb{Z})$$

where the last map is the first Chern class  $c_1$ . The quotient

$$\mathrm{Pic}^0 X := H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z})$$

is the group of topologically trivial complex line bundles and is a compact complex torus (in fact, an abelian variety). Note that if  $H^1(X, \mathbb{Z})$  vanishes, then  $\mathrm{Pic}^0 X$  vanishes and the first Chern class  $c_1 : \mathrm{Pic} X \rightarrow H^2(X, \mathbb{Z})$  is injective.

If  $H^1(X, \mathbb{Z})$  vanishes, it follows from the Universal Coefficient Theorem that the torsion subgroup of  $H^2(X, \mathbb{Z})$  is  $\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$ . Every torsion element of  $H^2(X, \mathbb{Z})$  is the Chern class of a holomorphic line bundle as a homomorphism  $\chi : H_1(X, \mathbb{Z}) \rightarrow \mathbb{C}^*$  gives rise to a flat (and therefore holomorphic) line bundle over  $X$ .

**Exercise 14.1.** Suppose that  $X$  is a compact Kähler manifold with  $H^1(X, \mathbb{Z}) = 0$ . Show that if  $\mathcal{L}$  is the flat line bundle over whose monodromy is given by the homomorphism  $\chi : H_1(X, \mathbb{Z}) \rightarrow \mathbb{C}^*$ , then

$$c_1(\mathcal{L}) = \chi \in \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*) \subseteq H^2(X, \mathbb{Z}).$$

These basic facts generalize to non-compact varieties. Suppose that  $X$  is a smooth quasi-projective variety. Define  $\mathrm{Pic} X$  to be the group of isomorphism classes of algebraic line bundles over  $X$ , and  $\mathrm{Pic}^0 X$  to be the kernel of the Chern class mapping

$$c_1 : \mathrm{Pic} X \rightarrow H^2(X, \mathbb{Z}).$$

**Theorem 14.2.** *Suppose that  $X$  is a smooth quasi-projective variety. If  $H^1(X, \mathbb{Q})$  vanishes, then  $\mathrm{Pic}^0 X = 0$  and the torsion subgroup of  $\mathrm{Pic} X$  is naturally isomorphic to  $\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$ .*

*Sketch of Proof.* There are several ways to prove this. One is to use Deligne cohomology which gives a Hodge theoretic computation of  $\text{Pic } X$ . Details can be found in [13], for example. A more elementary approach goes as follows. First pick a smooth compactification  $\overline{X}$  of  $X$ . Each line bundle  $\mathcal{L}$  over  $X$  can be extended to a line bundle over  $\overline{X}$ . Any two extensions differ by twists by the divisors in  $X$  that lie in  $\overline{X} - X$ . After twisting by suitable boundary components, we may assume that the extended line bundle also has trivial  $c_1$  in  $H^2(\overline{X}, \mathbb{Z})$ . (Prove this using the Gysin sequence.) It therefore gives an element of  $\text{Pic}^0 \overline{X}$ , which, by the discussion at the beginning of the section, is flat. This implies that the original line bundle  $\mathcal{L}$  over  $X$  is also flat. But if  $H^1(X, \mathbb{Q})$  vanishes, then  $H_1(X, \mathbb{Z})$  is torsion and  $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$  is the corresponding character  $\chi : H_1(X, \mathbb{Z}) \rightarrow \mathbb{C}^*$ . But since  $c_1(\mathcal{L})$  is trivial, this implies that  $\chi$  is the trivial character and that  $\mathcal{L}$  is trivial.  $\square$

This result extends to the orbifold case. By a smooth quasi-projective orbifold, we mean an orbifold  $\Gamma \backslash X$  where  $\Gamma$  has a subgroup  $\Gamma'$  of finite index that acts fixed point freely on  $X$  and where  $\Gamma' \backslash X$  is a smooth quasi-projective variety. There is a Chern class mapping

$$c_1 : \text{Pic}_{\text{orb}}(\Gamma \backslash X) \rightarrow H^2(\Gamma, \mathbb{Z}).$$

(The Picard group is constructed using equivariant algebraic line bundles on finite covers of  $\Gamma \backslash X$ .) The Chern class  $c_1$  can be constructed using the Borel construction, for example. Define  $\text{Pic}_{\text{orb}}^0(\Gamma \backslash X)$  to be the kernel of  $c_1$ .

**Theorem 14.3.** *Suppose that  $\Gamma \backslash X$  is a smooth quasi-projective orbifold. If  $H^1(X, \mathbb{Q})$  vanishes, then  $\text{Pic}_{\text{orb}}^0 X = 0$  and the torsion subgroup of  $\text{Pic } X$  is naturally isomorphic to  $\text{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{C}^*)$ .*

*Sketch of Proof.* Fix a finite orbifold covering of  $\Gamma' \backslash X$  of  $\Gamma \backslash X$  where  $\Gamma'$  is normal in  $\Gamma$  and acts fixed point freely on  $X$ . The Galois group of the covering is  $G = \Gamma/\Gamma'$ . Suppose that  $\mathcal{L}$  is an orbifold line bundle over  $\Gamma \backslash X$  with  $c_1(\mathcal{L}) = 0$  in  $H^2(\Gamma, \mathbb{Z})$ . This implies that the first Chern class of the pullback of  $\mathcal{L}$  to  $\Gamma' \backslash X$  is also trivial. Since this pullback has a natural  $G$ -action, this means that the corresponding point in  $\text{Pic}^0(\Gamma' \backslash X)$  is  $G$ -invariant. Since

$$H^1(\Gamma' \backslash X, \mathbb{Q})^G \cong H^1(\Gamma \backslash X, \mathbb{Q}) = 0$$

it follows that the  $G$ -invariant part of  $\text{Pic}(\Gamma' \backslash X)$  is finite. It follows that some power  $\mathcal{L}^{\otimes N}$  of the pullback of  $\mathcal{L}$  to  $\Gamma' \backslash X$  is trivial. This also

has a  $G$ -action. If  $s$  is a trivializing section of  $\mathcal{L}^{\otimes N}$ , then the product

$$\bigotimes_{g \in G} g \cdot s$$

is a  $G$ -invariant section of  $\mathcal{L}^{\otimes N|G|}$ . It follows that  $\mathcal{L}$  has a flat structure invariant under the  $G$ -action. But since  $\mathcal{L}$  has trivial  $c_1$ , it must have trivial monodromy. It is therefore trivial. It follows that  $\text{Pic}_{\text{orb}}(\Gamma \backslash X)$  is trivial.  $\square$

Assembling the pieces, we have:

**Corollary 14.4.** *If  $H^1(\Gamma_g, \mathbb{Q})$  vanishes, then the Chern class mapping*

$$\text{Pic}_{\text{orb}} \mathcal{M}_g \rightarrow H^2(\Gamma_g, \mathbb{Z})$$

*is injective.*  $\square$

## 15. RELATIONS IN $\Gamma_g$

In this section we write down some well known relations that hold in various mapping class groups. These will be enough to compute  $H_1(\Gamma_g)$ , which we shall do in the next section.

First some notation. We shall let  $S$  denote any compact oriented surface with (possibly empty) boundary. The corresponding mapping class group is defined to be

$$\Gamma_S := \pi_0 \text{Diff}^+(S, \partial S).$$

That is,  $\Gamma_S$  consists of isotopy classes of orientation preserving diffeomorphisms that act trivially on the boundary  $\partial S$  of  $S$ .

**Exercise 15.1.** Show that elements of  $\Gamma_S$  can be represented by diffeomorphisms that equal the identity in a neighbourhood of  $\partial S$ .

**Exercise 15.2.** Show that if  $S$  is a compact oriented surface and  $T$  is a compact subsurface, then there is a natural homomorphism  $\Gamma_T \rightarrow \Gamma_S$  obtained by extending elements of  $\Gamma_T$  to be the identity outside  $T$ .

An important special case is where we take  $T$  to be the cylinder  $[0, 1] \times S^1$  (with the product orientation). One element of  $\Gamma_T$  is the diffeomorphism

$$\tau : (t, \theta) \mapsto (t, \theta + 2\pi t).$$

**Theorem 15.3.** *If  $T$  is the cylinder, then  $\Gamma_T$  is infinite cyclic and is generated by  $\tau$ .*  $\square$

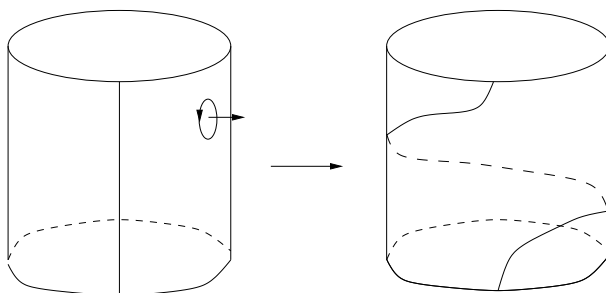


FIGURE 4. a positive Dehn twist

Now, if  $S$  is any surface, and  $A$  is a (smoothly imbedded) simple closed curve (SCC), then  $A$  has a neighbourhood that is diffeomorphic to the cylinder  $T = [0, 1] \times S^1$  as in Figure 5. Denote the image of  $\tau$  under the homomorphism  $\Gamma_T \rightarrow \Gamma_S$  by  $t_A$ . It is called the *Dehn twist about  $A$* .

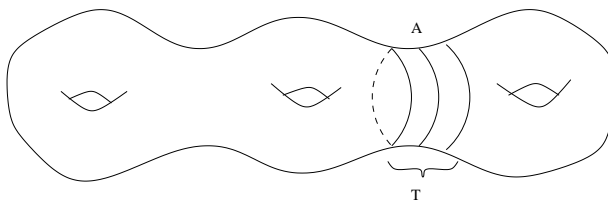


FIGURE 5.

**Theorem 15.4.** *The class of  $t_A$  in  $\Gamma_S$  is independent of the choice of the tubular neighbourhood  $T$  of  $A$  and depends only on the isotopy class of  $A$ . Every mapping class group is generated by Dehn twists.*  $\square$

**Exercise 15.5.** Show that if  $\phi \in \Gamma_S$  and  $A$  is a SCC in  $S$ , then

$$t_{\phi(A)} = \phi t_A \phi^{-1}.$$

Observe that if two elements of  $\Gamma_S$  can be represented by diffeomorphisms with disjoint support, then they commute. In particular:

**Proposition 15.6.** *If  $A$  and  $B$  are disjoint SCCs on  $S$ , then  $t_A$  and  $t_B$  commute in  $\Gamma_S$ .*  $\square$

**Exercise 15.7.** Show that if  $S$  is a surface with boundary and  $A$  and  $B$  are each non-separating SCCs in  $S$ , then there is an orientation preserving diffeomorphism  $\phi$  of  $S$  that takes one onto the other. Deduce that  $t_A$  and  $t_B$  are conjugate in  $\Gamma_S$ . Give an example to show that not all Dehn twists about bounding SCCs are conjugate in  $\Gamma_S$  when  $g \geq 3$ .

Next we consider what happens when  $A$  and  $B$  are two simple closed curves in  $S$  that meet transversally in one point as in Figure 6.

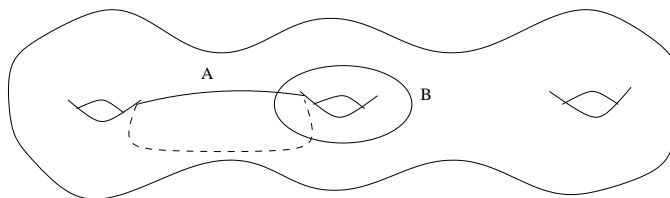


FIGURE 6.

**Exercise 15.8.** Show that if  $A$  and  $B$  are two SCCs in  $S$  that meet transversally in one point, then there is a neighbourhood of their union that is a compact genus 1 surface with one boundary component. Hint: Compute the homology of a small regular neighbourhood of the union and then apply the classification of compact oriented surfaces.

So any relation that holds between Dehn twists such curves in a genus one surface with one boundary component will hold in all surfaces. Let  $S$  be a compact genus 1 surface with one boundary component and let  $A$  and  $B$  be the two SCCs in the diagram:

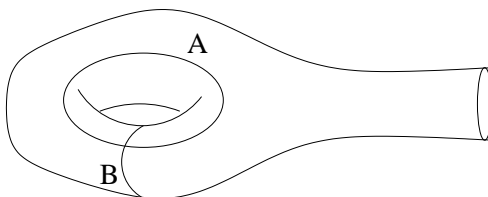


FIGURE 7.

**Theorem 15.9.** *With notation as above, we have*

$$t_A t_B t_A = t_B t_A t_B$$

*in  $\Gamma_S$  and therefore in all mapping class groups.*

This relation is called the *braid relation* as it comes from the braid group on 3 strings<sup>9</sup> using the following technique.

Denote the unit disk by  $D$ . We view this as a manifold with boundary. Suppose that  $P$  is a set of  $n$  distinct points of  $D$ , none of which lies on  $\partial D$ . The braid group  $B_n$  is defined to be

$$B_n := \pi_0 \text{Diff}^+(D, (P)),$$

<sup>9</sup>A good reference for braid groups is Joan Birman's book [4].

where  $\text{Diff}^+(D, (P))$  denotes the group of orientation preserving diffeomorphisms of  $D$  that fix  $P$  as a set, but may permute its elements. There is a surjective homomorphism

$$B_n \rightarrow \text{Aut } P \cong \Sigma_n$$

onto the symmetric group on  $n$  letters.

Suppose that  $U$  is a disk imbedded in  $D$  such that

$$\partial U \cap P = \emptyset$$

and  $U \cap P$  consists of two distinct points  $x$  and  $y$  of  $P$ . Then there is an element  $\sigma_U$  of  $B_n$  whose square is the Dehn twist  $t_{\partial U}$  about the boundary of  $U$  and which swaps  $x$  and  $y$ .

It can be represented schematically as in Figure 8: The braid group

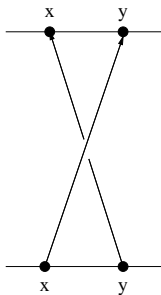


FIGURE 8. a basic braid

$B_n$  is generated by the braids  $\sigma_1, \dots, \sigma_{n-1}$  illustrated in Figure 9. Note

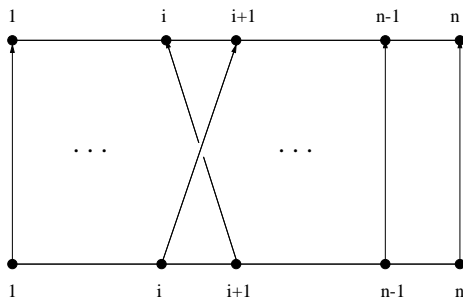


FIGURE 9. the generator  $\sigma_i$

that  $\sigma_i$  and  $\sigma_j$  commute when  $|i - j| > 1$ , and that

$$(6) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Now suppose that  $(S, \partial S) \rightarrow (D, \partial D)$  is a branched covering, unramified over the boundary. Suppose that the image of the branch points

is  $P$ . Then there is a natural homomorphism

$$B_n \rightarrow \Gamma_S.$$

So relations in  $B_n$  will give relations in  $\Gamma_S$ . The relations we are interested in come from certain double branched coverings of the disk.

Suppose that  $(S, \partial S) \rightarrow (D, \partial D)$  is a 2-fold branched covering. The inverse image of a smooth arc  $\alpha$  joining two critical values  $p_1, p_2 \in P$  and avoiding  $P$  otherwise, is an SCC, say  $A$ , in  $S$ . There is a small neighbourhood  $U$  of  $\alpha$  that is diffeomorphic to a disk and whose intersection with  $P$  is  $\{p_1, p_2\}$  as in Figure 10. There is a braid  $\sigma$  supported in  $U$  that swaps  $p_1$  and  $p_2$  by rotating in the positive direction about  $\alpha$ .

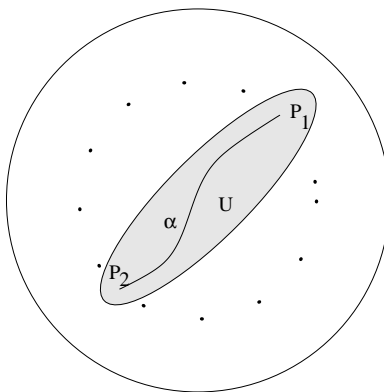


FIGURE 10.

**Proposition 15.10.** *The image of  $\sigma$  under the homomorphism  $B_n \rightarrow \Gamma_S$  is the Dehn twist  $t_A$  about  $A$ .*  $\square$

**Exercise 15.11.** Show that a genus 1 surface  $S$  with one boundary component can be realized as a 2:1 covering of  $D$ , branched over 3 points.

We therefore have a homomorphism  $B_3 \rightarrow \Gamma_S$ . Note that the inverse image of the two arcs  $\alpha$  and  $\beta$  in Figure 11 under the covering of the disk branched over  $\{p_1, p_2, p_3\}$  is a pair of SCCs in  $S$  that intersect transversally in one point. The braid relation in  $\Gamma_S$  now follows as we have the braid relation (6)

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$

in  $B_3$ .

More relations can be obtained this way. Suppose that  $S$  is a compact genus 1 surface with 2 boundary components.

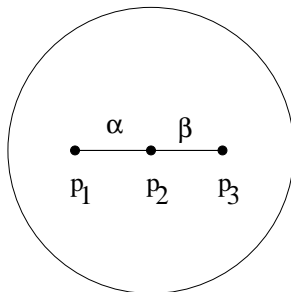


FIGURE 11.

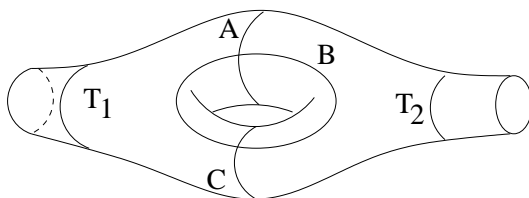


FIGURE 12.

Let  $A$ ,  $B$ ,  $C$ ,  $T_1$  and  $T_2$  be the SCCs in Figure 12. Denote the Dehn twists about  $A$ ,  $B$  and  $C$  by  $a$ ,  $b$  and  $c$ , and those about  $T_1$  and  $T_2$  by  $t_1$  and  $t_2$ .

**Theorem 15.12.** *The relation*

$$(abc)^4 = t_1 t_2$$

*holds in the mapping class group  $\Gamma_S$ .*

**Corollary 15.13.** *If  $S$  is a compact genus 1 surface with one boundary component, then the relation*

$$(ab)^6 = t$$

*holds in  $\Gamma_S$ , where  $a$  and  $b$  denote Dehn twists about a pair of SCCs that intersect transversally in one point and  $t$  denotes a Dehn twist about the boundary.*

**Exercise 15.14.** Deduce Corollary 15.13 from Theorem 15.12 by capping off one boundary component and using the braid relation, Theorem 15.9.

Theorem 15.12 is proved in the following exercise.

**Exercise 15.15.** Suppose that  $S$  is a genus 1 surface with two boundary components. Show that  $S$  is a double covering of the disk, branched



over 4 points. Use this to construct a homomorphism from the braid group  $B_4$  on 4 strings into  $\Gamma_S$ . Use this to prove the relation

$$(abc)^4 = t_1 t_2$$

where  $a, b, c, t_1$  and  $t_2$  denote Dehn twists on the SCCs  $A, B, C, T_1$  and  $T_2$  in the diagram above. Note that in the braid group, we have the relation

$$t = \sigma_1 \sigma_2 \sigma_3$$

where  $t$  is Dehn twist about the boundary of the disk and  $\sigma_i$  is the  $i$ th standard generator of  $B_4$ .

There is one final relation. It is called the *lantern relation* and is due to Johnson and Harer independently. Let  $S$  be a disk with 3 holes. (That is, a genus 0 surface with 4 boundary components.) Consider the SCCs in Figure 13.

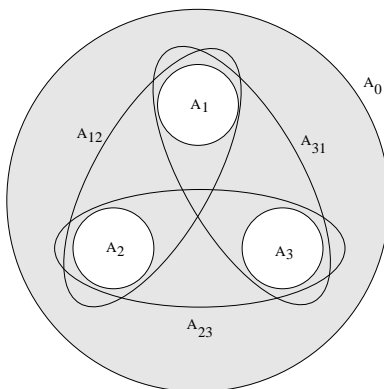


FIGURE 13. the lantern configuration

**Theorem 15.16.** *The relation*

$$a_0 a_1 a_2 a_3 = a_{12} a_{23} a_{31}$$

*holds in  $\Gamma_S$  where  $a_i$  denotes the Dehn twist about  $A_i$  and  $a_{ij}$  denotes the Dehn twist about  $A_{ij}$ .  $\square$*

## 16. COMPUTATION OF $H_1(\Gamma_g, \mathbb{Z})$

The fact  $\Gamma_g$  is generated by Dehn twists and the relations given in the previous section allow the computation of  $H_1(\Gamma_g)$ .

**Theorem 16.1** (Harer). *If  $g \geq 1$ , then*

$$H_1(\Gamma_g, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/12\mathbb{Z} & g = 1; \\ \mathbb{Z}/10\mathbb{Z} & g = 2; \\ 0 & g \geq 3. \end{cases}$$

*Proof.* We begin with the observation that if  $S$  is a compact oriented genus  $g$  surface, then all Dehn twists on non-separating SCCs lie in the same homology class as they are conjugate by Exercise 15.5. Denote their common homology class by  $L$ . Next, using the relations coming from an imbedded genus 1 surface with one boundary component, we see that the homology class of any separating SCC that divides  $S$  into a genus 1 and genus  $g - 1$  surfaces has homology class  $12L$ . Using the relation that comes from an imbedded genus 1 surface with 2 boundary components, we see that the homology class of every separating SCC is an integer multiple of  $L$ . It follows that  $H_1(\Gamma_g)$  is cyclic and generated by  $L$ .

Now suppose that  $g \geq 3$ . Then we can find an imbedded lantern as in Figure 14. Since each of the curves in this lantern is non-separating,

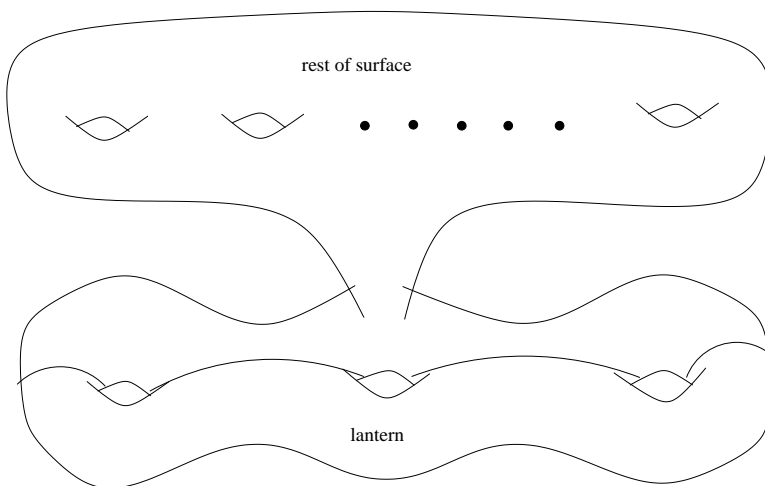


FIGURE 14.

the lantern relation tells us that  $3L = 4L$ , which implies that  $L = 0$ . This proves the vanishing of  $H_1(\Gamma_g)$  when  $g \geq 3$ .

When  $g = 1$ , the relation for a genus 1 surface with one boundary component implies that  $12L = 0$  as the twist about the boundary is trivial in  $\Gamma_1$ . Thus  $H_1(\Gamma_1)$  is a quotient of  $\mathbb{Z}/12\mathbb{Z}$ . But, as we shall

explain a little later, the fact that  $\text{Pic}_{\text{orb}} \mathcal{M}_1$  is at least as big as  $\mathbb{Z}/12$  implies that  $H_1(\Gamma_1) = \mathbb{Z}/12\mathbb{Z}$ .<sup>10</sup>

A genus 2 surface can be obtained from a genus 1 surface with two boundary components by identifying the boundary components. The relation obtained for a genus 1 surface with 2 boundary components gives  $12L = 2L$ , so that  $10L = 0$ . This shows that  $H_1(\Gamma_2)$  is a quotient of  $\mathbb{Z}/10\mathbb{Z}$ . But, as in the genus 1 case, the theory of Siegel modular forms shows that it cannot be any smaller. So we have  $H_1(\Gamma_2) = \mathbb{Z}/10\mathbb{Z}$ .  $\square$

**Corollary 16.2.** *For all  $g \geq 1$ ,  $H^1(\Gamma_g, \mathbb{Z})$  vanishes.*  $\square$

### 17. COMPUTATION OF $\text{Pic}_{\text{orb}} \mathcal{M}_g$

Since  $H^1(\Gamma_g, \mathbb{Q})$  is torsion, it follows from Corollary 14.4 that

$$c_1 : \text{Pic}_{\text{orb}} \mathcal{M}_g \rightarrow H^2(\Gamma_g, \mathbb{Z})$$

is injective. Since  $H_1(\Gamma_g)$  is torsion, the Universal Coefficient Theorem implies that the sequence

$$0 \rightarrow \text{Hom}(H_1(\Gamma_g, \mathbb{Z}), \mathbb{C}^*) \rightarrow H^2(\Gamma_g, \mathbb{Z}) \rightarrow \text{Hom}(H_2(\Gamma_g), \mathbb{Z}) \rightarrow 1$$

is exact. To determine the rank of  $H^2(\Gamma_g)$ , we need the following fundamental and difficult result of Harer [15].

**Theorem 17.1.** *The rank of  $H_2(\Gamma_g, \mathbb{Q})$  is 0 if  $g \leq 2$  and 1 if  $g \geq 3$ .*  $\square$

Combining this with our previous discussion, we have:

**Theorem 17.2.** *If  $g \geq 1$ , then*

$$H^2(\Gamma_g, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/12\mathbb{Z} & g = 1; \\ \mathbb{Z}/10\mathbb{Z} & g = 2; \\ \mathbb{Z} & g \geq 3. \end{cases}$$

There is one obvious orbifold line bundle over  $\mathcal{M}_g$ . Namely, if we take the universal curve  $\pi : \mathcal{C} \rightarrow \mathcal{M}_g$  over the orbifold  $\mathcal{M}_g$ , then we can form the line bundle

$$\mathcal{L} := \det \pi_* \omega_{\mathcal{C}/\mathcal{M}_g}$$

over  $\mathcal{M}_g$ , where  $\omega_{\mathcal{C}/\mathcal{M}_g}$  denotes the relative dualizing sheaf. The fiber over  $[C] \in \mathcal{M}_g$  is

$$\Lambda^g H^0(C, \Omega_C^1).$$

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<sup>10</sup>The abelianization of  $SL_2(\mathbb{Z})$  can also be computed using, for example, the presentation of  $PSL_2(\mathbb{Z})$  given in [24].

**Theorem 17.3.** *If  $g \geq 1$ , the orbifold Picard group of  $\mathcal{M}_g$  is cyclic, generated by  $\mathcal{L}$  and given by*

$$\mathrm{Pic}_{\mathrm{orb}} \mathcal{M}_g \cong \begin{cases} \mathbb{Z}/12\mathbb{Z} & g = 1; \\ \mathbb{Z}/10\mathbb{Z} & g = 2; \\ \mathbb{Z} & g \geq 3. \end{cases}$$

*Proof.* All but the generation by  $\mathcal{L}$  follows from preceding results. In genus 1 and 2, the generation by  $\mathcal{L}$  follows from the theory of modular forms.

Suppose that  $g \geq 3$ . Denote the first Chern class of  $\mathcal{L}$  by  $\lambda$ . To prove that  $\mathcal{L}$  generates  $\mathrm{Pic}_{\mathrm{orb}} \mathcal{M}_g$ , it suffices to show that  $\lambda$  generates  $H^2(\Gamma_g, \mathbb{Z})$ . The following proof of this I learned from Shigeyuki Morita. It assumes some knowledge of characteristic classes. A good reference for this topic is the book [21] by Milnor and Stasheff.

We begin by recalling the definition of the signature of a compact oriented 4-manifold. Every symmetric bilinear form on a real vector space can be represented by a symmetric matrix. The signature of a non-degenerate symmetric bilinear form is the number of positive eigenvalues of a representing matrix minus the number of negative eigenvalues. To each compact oriented 4-manifold  $X$ , we associate the symmetric bilinear form

$$H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \rightarrow \mathbb{R}$$

defined by

$$\xi_1 \otimes \xi_2 \mapsto \int_X \xi_1 \wedge \xi_2.$$

Poincaré duality implies that it is non-degenerate. The signature  $\tau(X)$  of  $X$  is defined to be the signature of this bilinear form. It is a cobordism invariant.

The Hirzebruch Signature Theorem (see [21, Theorem 19.4], for example) asserts that

$$\tau(X) = \frac{1}{3} \int_X p_1(X)$$

where  $p_1(X) \in H^4(X, \mathbb{Z})$  is the first Pontrjagin class of  $X$ . When  $X$  is a complex manifold,

$$(7) \quad p_1(X) = c_1(X)^2 - 2c_2(X).$$

Now suppose that  $X$  is a smooth algebraic surface and that  $T$  is a smooth algebraic curve. Suppose that  $\pi : X \rightarrow T$  is a family whose

fibers are smooth curves of genus  $g \geq 3$ .<sup>11</sup> Denote the relative cotangent bundle  $\omega_{X/T}$  of  $\pi$  by  $\omega$ . Then it follows from the exact sequence

$$0 \rightarrow \pi^* \Omega_S^1 \rightarrow \Omega_X^2 \rightarrow \omega_{X/T} \rightarrow 0$$

that

$$c_1(X) = \pi^* c_1(T) - c_1(\omega) \text{ and } c_2(X) = -c_1(\omega) \wedge \pi^* c_1(T).$$

Plugging these into (7) we see that  $p_1(X) = c_1(\omega)^2$ . Using integration over the fiber, we have

$$\tau(X) = \frac{1}{3} \int_X c_1(\omega)^2 = \frac{1}{3} \int_T \pi_*(c_1(\omega)^2).$$

It is standard to denote  $\pi_*(c_1(\omega)^2)$  by  $\kappa_1$ . Thus we have

$$\tau(X) = \frac{1}{3} \int_T \kappa_1.$$

An easy consequence of the Grothendieck-Riemann-Roch Theorem is that  $\kappa_1 = 12\lambda$ . This is proved in detail in the book of Harris and Morrison [17, pp. 155–156]. It follows that for a family  $\pi : X \rightarrow T$  of smooth curves

$$\tau(X) = 4 \int_T \lambda.$$

The last step is topological. Suppose that  $F$  is a compact oriented surface and that  $p : W \rightarrow F$  is an oriented surface bundle over  $F$  where the fibers of  $p$  are compact oriented surfaces of genus  $g \geq 3$ . Denote the local system of the first integral homology groups of the fibers by  $\mathbb{H}$ . There is a symmetric bilinear form

$$(8) \quad H^1(F, \mathbb{H}_{\mathbb{R}}) \otimes H^1(F, \mathbb{H}_{\mathbb{R}}) \rightarrow \mathbb{R}$$

obtained from the cup product and the intersection form. Poincaré duality implies that it is non-degenerate. It follows from the Leray-Serre spectral sequence of  $p$  that

$$\tau(W) = - \text{the signature of the pairing (8)}.$$

The local system  $\mathbb{H}$  over  $F$  corresponds to a mapping  $\phi : F \rightarrow BSp_g(\mathbb{Z})$  into the classifying space of the symplectic group. Meyer [20] shows that there is a cohomology class  $m \in H^2(Sp_g(\mathbb{Z}), \mathbb{Z})$  whose value on  $\phi$  is the signature of the pairing (8). It follows from this and the discussion above, that under the mapping

$$\rho^* : H^2(Sp_g(\mathbb{Z}), \mathbb{Z}) \rightarrow H^2(\Gamma_g, \mathbb{Z})$$

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<sup>11</sup>Such families exist — see [17, p. 45, p. 55].

induced by the canonical homomorphism  $\rho : \Gamma_g \rightarrow Sp_g(\mathbb{Z})$ ,  $m$  goes to  $-4\lambda$ . Mayer also shows that the image of

$$m : H^2(\Gamma_g, \mathbb{Z}) \rightarrow \mathbb{Z}$$

is exactly  $4\mathbb{Z}$ , which implies that  $\lambda$  generates  $H^2(\Gamma_g, \mathbb{Z})$ .  $\square$

As a corollary of the proof, we have:

**Corollary 17.4.** *For all  $g \geq 3$ , both  $H^2(Sp_g(\mathbb{Z}), \mathbb{Z})$  and  $H^2(\Gamma_g, \mathbb{Z})$  are generated by  $\lambda$  and the natural mapping*

$$\rho^* : H^2(Sp_g(\mathbb{Z}), \mathbb{Z}) \rightarrow H^2(\Gamma_g, \mathbb{Z})$$

*is an isomorphism.*  $\square$

Note that  $H^2(Sp_2(\mathbb{Z}), \mathbb{Z})$  is infinite cyclic, while  $H^2(\Gamma_2, \mathbb{Z})$  is cyclic of order 10. So  $\rho^*$  is not an isomorphism in genus 2.

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